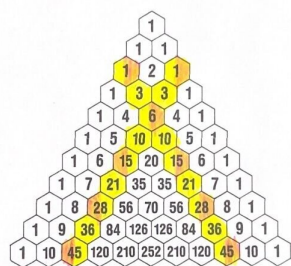
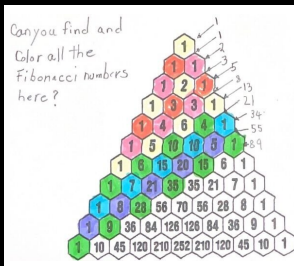


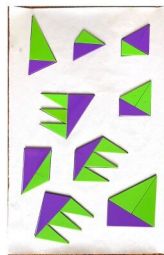
Math Circle 2023-2024 Cyrus, Marc, Dar



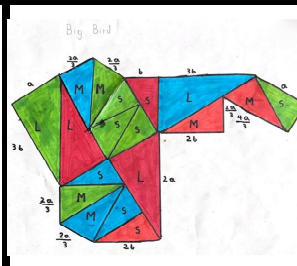
1. Visual Patterns in Pascal's Triangle



2. Visual Patterns Continued



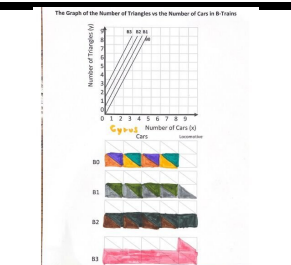
3. Fitz D Pieces Algebra Exploration



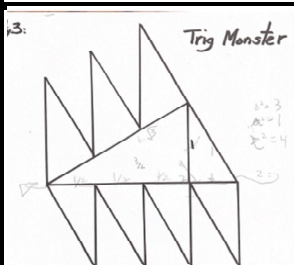
4. Fitz D Pieces Continued

	S	M	L
A	A=2S	A=1M	A=3L
B	B=3S	B=2M	B=4L
C	C=6S	C=3M	C=2L
D	D=8S	D=6M	D=3L
E	E=10S	E=8M	E=5L
F	F=18S	F=12M	F=6L
G	G=24S	G=18M	G=9L
	G=24S	L=3S	L=3L
	G=8L	S=1/3L	
	G=24S	M=2S	M=2L
	G=8L	S=1/3M	
	G=24S	L=3M	L=3L
	G=8L	M=1/3L	

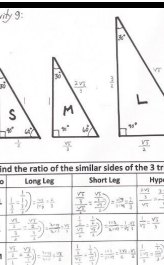
5. Horation FD Rectangle Ratio Exploration



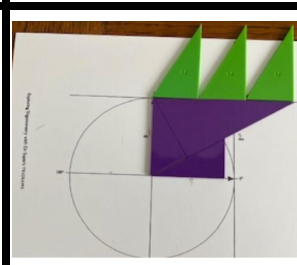
6.D. A-Train - Line & Slope Investigation



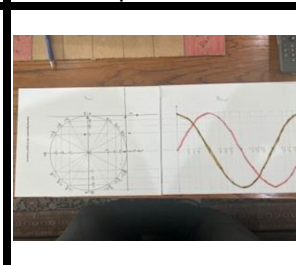
7.Exploring Trigonometry with Trigrams 1



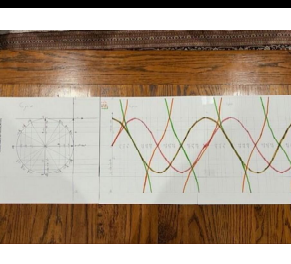
8.Exploring Trigonometry with Trigrams 2



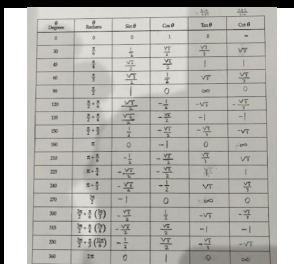
9.Exploring Trigonometry with Trigrams 3



10.SOC CAH TOA & Graphing Sin Cos 1



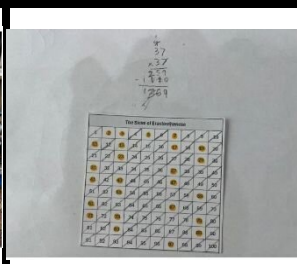
11.SOC CAH TOA Graphing Sin Cos 2



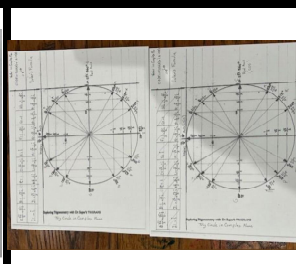
12.SOC CAH TOA Graphing Sin Cos 3



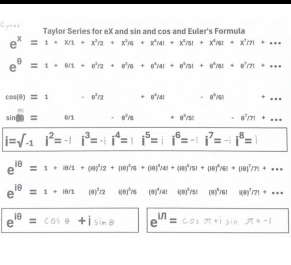
13.Blue1Brown Trigonometry



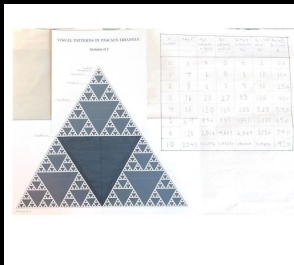
14.The Sieve of Eratosthenise



15.3Blue1Brown Complex Numbers



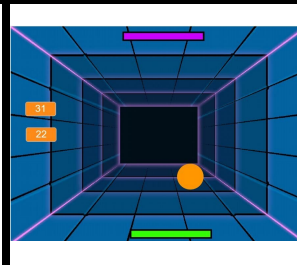
16. Euler's Formula & Identity



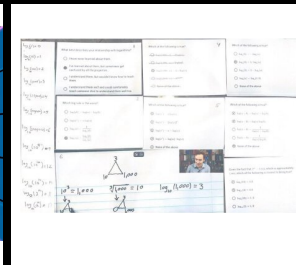
17. Pascal's Triangle & Divisibility Patterns 1



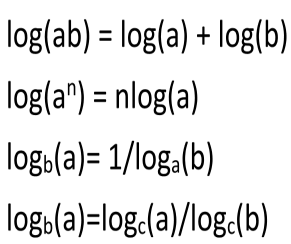
18. Pascal's Triangle & Divisibility Patterns 2



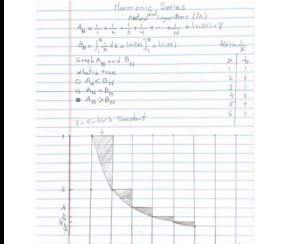
19.Scratch Programing



20. Logarithmths



21. Logarithmths



22.Logarithmths & Exponentials



23. Graduation and Celebration



23. Graduation and Celebration



23. Graduation and Celebration

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1	2023-08-16	Visual Patterns' in Pascal's Triangle
2	2022-08-23	Visual Patterns' Continued
3	2023-08-30	Fitz the Pieces Algebra Exploration
4	2022-09-13	Fitz the Pieces Algebra Exploration Continued
5	2023-11-01	Hratio FD REctangle - Ratio Exploration
6	2023-11-15	D. A-Train - Slope Investigation
7	2022-11-29	Exploring Trigonometry with Trigrams 1
8	2022-12-06	Exploring Trigonometry with Trigrams 2
9	2022-12-13	Exploring Trigonometry with Trigrams 3
10	2024-01-10	SOC CAH TOA and Graphing Sin and Cos 4
11	2024-01-24	SOC CAH TOA and Graphing Sin and Cos 5
12	2024-01-31	Exploring Trigonometry with Trigrams 6
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17	2024-04-03	Pascal's Triangle and Divisibility Patterns 1+ Euler 2
18	2024-04-10	Pascal's Triangle and Divisibility Patterns 2+Euler 3
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20	2024-04-24	Logarithms
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22	2024-05-08	Logarithms and Exponentials
23	2024-05-15	Graduation and Celebration

Math Circle 1 08-16-2023 2:00 to 3:00 pm

Cyrus, Marc and Dar

In this first session Of the new Academic year we reviewed and continued our work on Pascal's Triangle (Omar Khayam's Triangle). I used a couple of pages from the book [Visual Patterns in Pascaal's Triangle by Dale Seymour](#).

First, we looked at Figurate Number Sequences: Triangular, Square, Pentagonal and Hexagonal Sequences. They used the page that they had completed last year as a guide.

They found and colored the Triangular Numbers on the Pascal's Triangle. Then they also colored the hexagonal numbers. After some effort Cyrus recalled that sum of pairs of consecutive triangular numbers made the square numbers.

Next they found the powers of 2 and marked it on their triangles. We spent a good portion of the time in finding the powers of 11. They liked this much more and saw quickly that the powers of 11 are in the Pascal's Triangle. However, after the fifth power the numbers become double digit in each cell. I let them struggle a little but told them that they should think of the number's in the Pascal's triangle as being in a $1 \leq 10$ machine. Then they quickly went down to the last row even though the numbers were getting larger and larger. They both like these kinds of challenges very much.

Next I tried to relate the powers of $(x+y)$ to the elements of the pascals triangle and I had a lot less success. They were getting restless as they wanted to go play with the Ping Pong Ball machine! Never the lerss we went up to $(x+y)^5$ and they were recording it but stopped there as they were not really sure how the relation really worked. Of course working with symbols is harder for them than with numbers. I had made a table that could simplyfy the explanations but did not get to use it. At the end I tried to quickly relate the values in the Pascal's Triangle to the number of combinations of x and y but it was not getting across. We may look at this again some time in the future or in the next session.

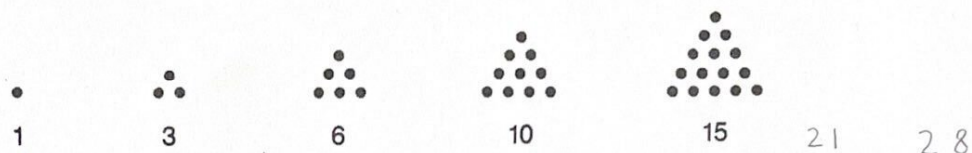
It was an OK session but they were clearly distracted by thinking about going and playing with the ball machine;=>

PS I did come up with idea of the Dr. Super's Pyramid and worked on it in the next couple of nights and I may try to discuss it with them. It turns out that this pyramid or Tetrahydron is well know and in Wikipedia but for me it was exciting to discover it by myself.

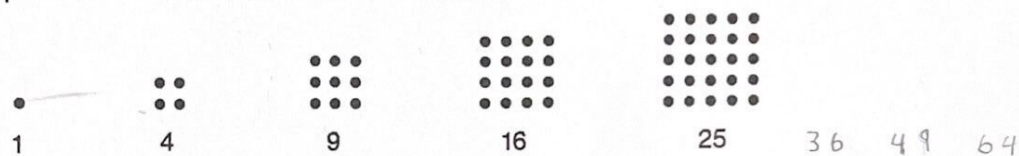
INTRODUCTION TO FIGURATE NUMBERS IN PASCAL'S TRIANGLE

The ancient Greeks were very interested in numbers that are connected by geometric forms. *Figurate numbers* (also known as polygonal numbers) are those numbers that can be arranged in a polygonal array.

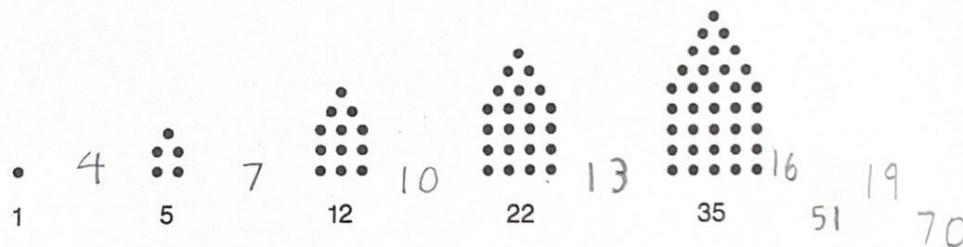
Triangular numbers can be represented by dots in a triangular array.



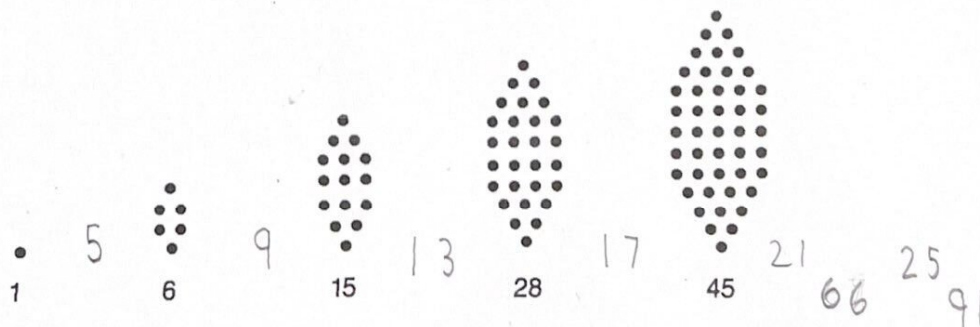
Square numbers can be arranged in a square array and are of the form n^2 where n equals the number of dots on a side.



Pentagonal numbers can be arranged in a pentagonal array.

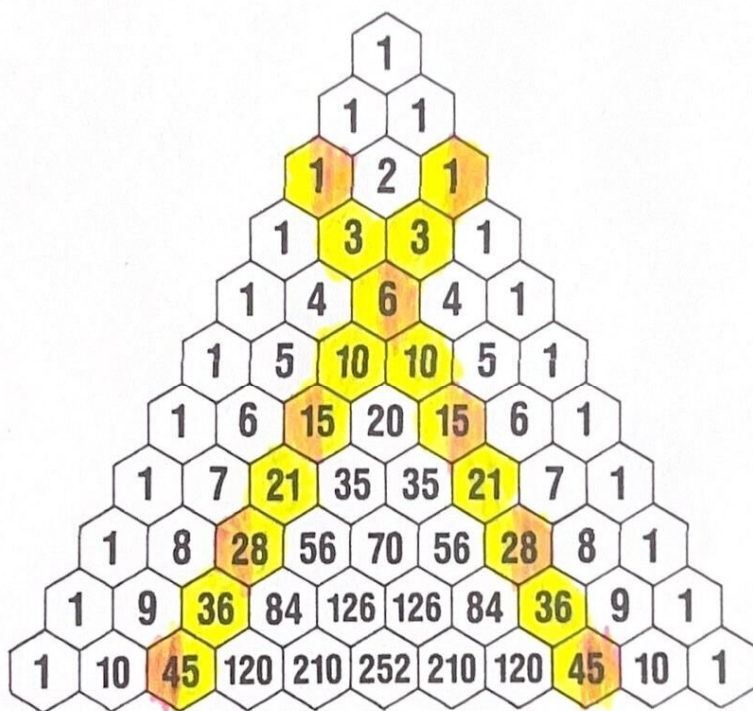


Hexagonal numbers can be shown as dots in an hexagonal array.

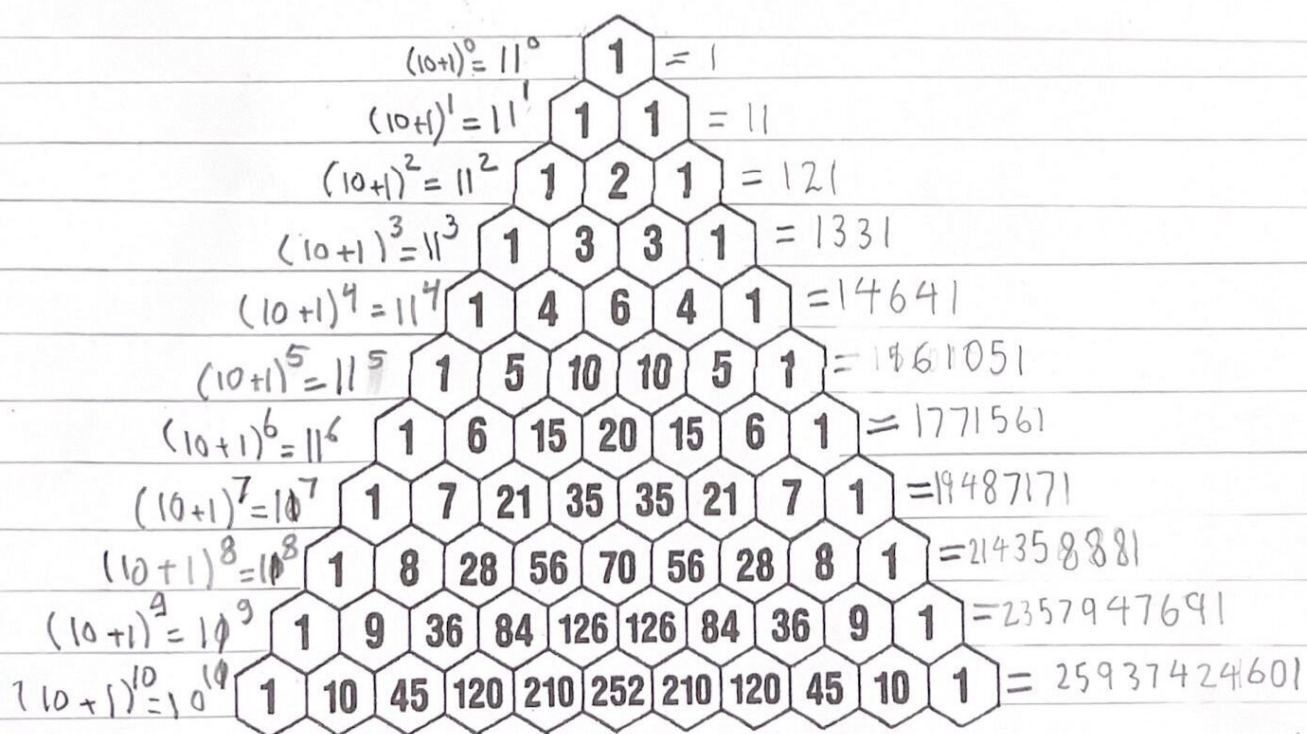


1. (a) What is the relationship between triangular numbers and square numbers?
- (b) Triangular and pentagonal numbers?
- (c) Triangular and hexagonal numbers?

POLYGONAL PATTERNS



1. The square numbers 1, 4, 9, 16, 25, 36, ... can be found in at least two patterns in Pascal's Triangle. What are these patterns? (Hint: The patterns involve the sum of certain numbers in the triangle.)
2. Can you find a pattern for pentagonal numbers? If so, what is it?
3. Look for the patterns for triangular numbers and hexagonal numbers. What do the patterns have in common?



$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^6 =$$

$$(x+y)^7 =$$

$$(x+y)^8 =$$

$$(x+y)^9 =$$

$$(x+y)^{10} =$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$(x+y)^0 = 1$$

$$(x+y)^1 = x+y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x+y)^6 =$$

$$(x+y)^7 =$$

$$(x+y)^8 =$$

$$(x+y)^9 =$$

$$(x+y)^{10} =$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Math Circle 1 08-16-2023 2:00 to 3:10 pm

Cyrus, Marc, and Dar

In this session we continued with patterns in Pascal's Triangle (Omar Khayam's Triangle). I used a few more of pages from the book [Visual Patterns in Pascaal's Triangle by Dale Seymour](#).

We continued our work with discovering and coloring Square Numbers then they found the Fibonacci Numbers and they also found the Pentagonal Numbers on Pascal's Triangles. Some of these in particular the Fibonacci numbers are not easy to find so I had colored a few of them to get them started.

Next, they saw that the elements of the Pascal's Triangle are the binomial coefficients. They started with the element in 4th row and 2nd column that I had colored for them and checked that it was $4!/2!2!$ then I asked them to find the binomial for (6,4). After some back and forth they both got 20 and found the value in the Pascal's Triangle in the 6th row and 3rd column. We discussed how every element was a binomial coefficient of this form. Cyrus asked about Dr. Super Pyramid that I had told him I had discovered and then found in Wikipedia, and we agreed to look at it at the end of the session.

Next, they looked for prime numbers in Pascal's Triangle. They found them in the first diagonals: 2, 3, 5, 7, 11, 13. They thought there were more inside the Triangle and I had them check again and they agreed that there were no other ones except on these two diagonals. Then they tried to find what was going on in the rows that had the prime and they found that all the numbers on these rows except for 1 were divisible by the prime on that row. They were both very impressed by this and I told them for all the rows in the Pascal's Triangle this would be the case. I saw this on YouTube last night and I was really impressed by it also so I had added it at the last minute.

Finally, I had them look at the Figurate numbers and some expressions involving them (see Table) and try to find what figurate number the expression corresponded to. This took some time as they are not very familiar with subscripts. So for some of them like $3T_{n-1} + T_n$ It was hard to see that $n-1$ was an index that had to be evaluated and then the value T_{n-1} was to be looked up and finally multiplied by 3. I gave them as little help as possible and let them figure it out mostly by themselves but in the end they were able to complete all the work.

By now the hour had gone by and a few more minutes so I did not start the last activity of linking $(x+y)^n$ to the elements of the Pascal's Triangle (which I will do next time) instead we built a Pyramid with tennis balls and I was trying to tell them about the Dr. Super Pyramid when Mihai and then Aryan came and the boys ran to play with the Ping Pong Ball machine.

It was a great session and they both enjoyed the activities and were impressed by some of them. Pascal's Triangle is such a rich topic for numbers, sequences and patterns and Cyrus and Marc both love numbers and patterns in particular.

Challenge Question

If P is a odd prime number prove that $(P-1)(P)(P+1)$ is always divisible by 24

For example if $P=3$ then $P-1=2$ and $P+1=4$ so $2 \times 3 \times 4 = 24$

You may want to try a few smaller primes like 5 or 7 first then try to find a proof.

$$P = 19$$

$$19 \times 20 \times 18 =$$

$$6,840$$

$P-1$ even

or

$P+1$ even

is

divisible

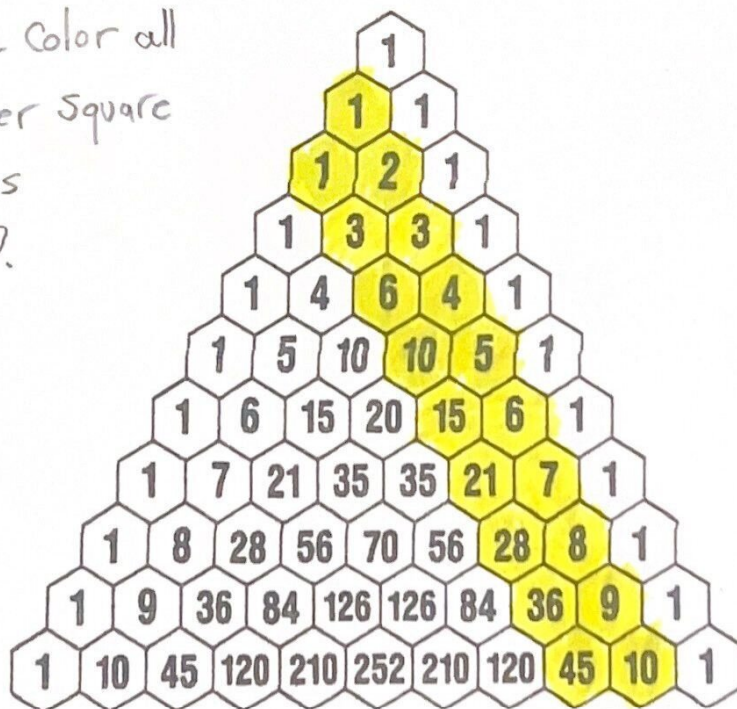
by

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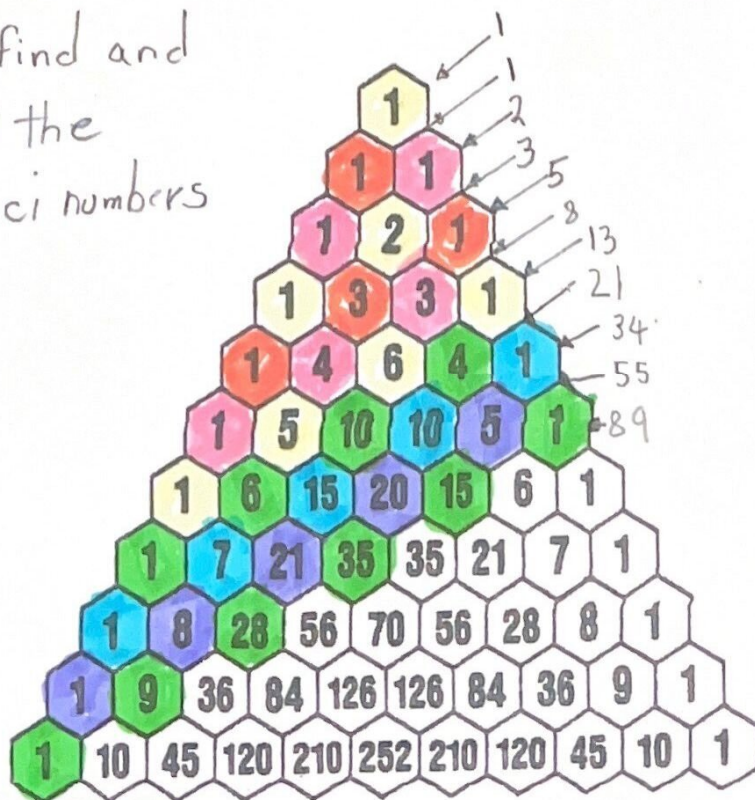
$$10 \times 11 \times 12$$

We concluded that we get $2 \times 3 \times 4$
 $\times n$ so we have $24n$

Can you color all
the other square
numbers
here?

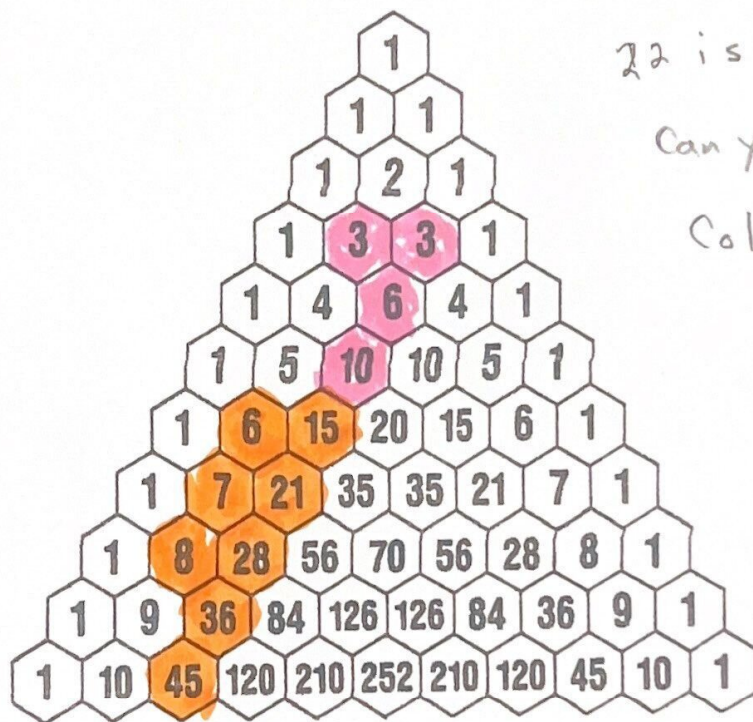


Can you find and
color all the
Fibonacci numbers
here?



Pentagonal numbers: 12, 22, 35, 51, 70, 92, 117, ...

$\underbrace{12}_{10} \quad \underbrace{22}_{13} \quad \underbrace{35}_{16} \quad \underbrace{51}_{19} \quad \underbrace{70}_{22} \quad \underbrace{92}_{25} \quad \dots$



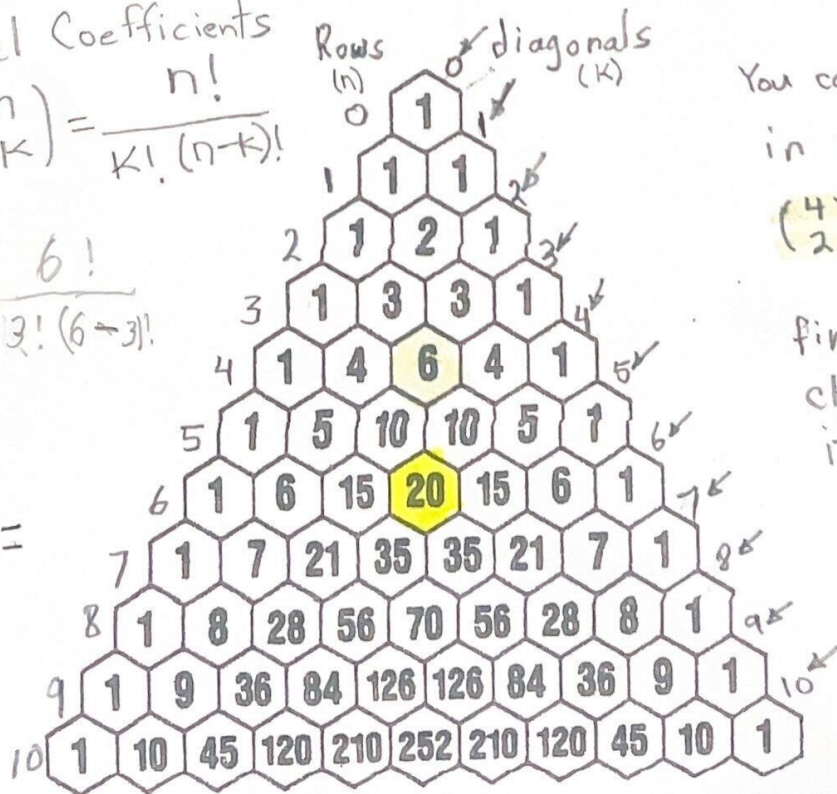
22 is colored here
Can you find and
color 70 and 117?

Binomial Coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{6}{3} = \frac{6!}{3!(6-3)!}$$

$$\frac{6!}{3! \cdot 3!} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$$



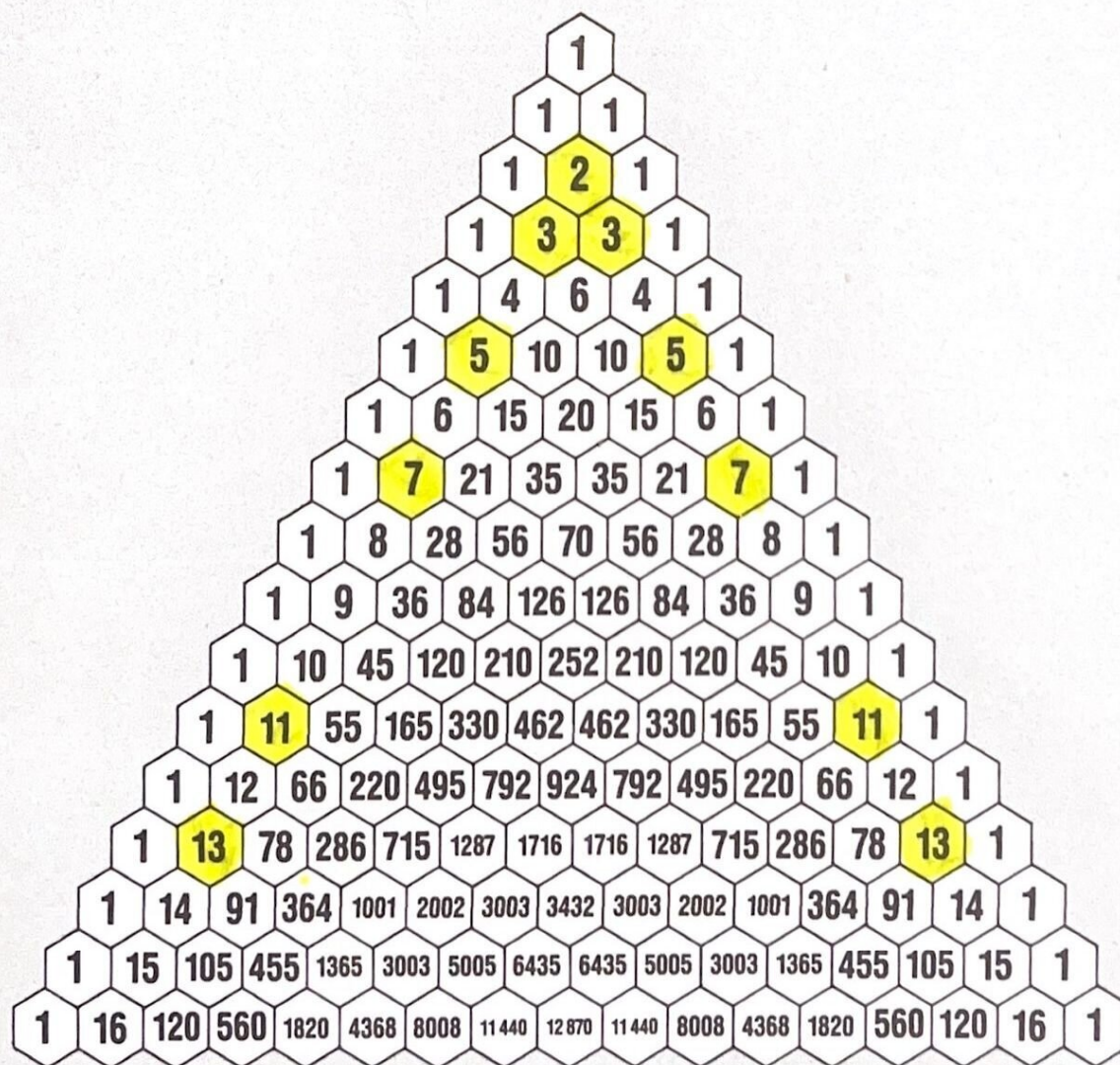
You can find $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
in n^{th} row and k^{th} diagonal

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 2} = 1$$

find the value and
check it and color
it for:

$$\binom{7}{2} \quad \binom{8}{2} \quad \binom{5}{3}$$

PASCAL'S TRIANGLE



Find and Color all the
Primes.
What do you notice?

Figurate Numbers		n						
		1	2	3	4	5	6	7
Triangular Numbers	T_n	1	3	6	10	15	21	28
Square Numbers	S_n	1	4	9	16	25	36	49
Pentagonal Numbers	P_n	1	5	12	22	35	50	66
Hexagonal Numbers	H_n	1	6	15	28	45	66	89

Evaluate each one of the following Expressions and find which one of the Figurate Numbers it represents

		n						
		1	2	3	4	5	6	7
$T_{n-1} + T_n$	S_n	1	$\frac{1+3}{4}$	$\frac{3+6}{9}$				
T_{2n-1}	h_n	1	$\frac{6-1}{2}$	15				
$T_{n-1} + S_n$	p_n	1	5	12				
$3T_{n-1} + T_n$	h_n	1	$\frac{3+3}{6}$	$\frac{9+6}{15}$	$\frac{18+10}{28}$			
$T_{n-2} + T_{n-1} + 2n - 1$	S_n	1	$\frac{1+3}{4}$	$\frac{1+3+5}{9}$				

Math Circle 3 08-30-2023 1:55 to 3:05 pm

Cyrus, Marc, and Dar

We started a new activity that is called Fitz the Pieces with my Dr. Super Triangles Manipulative and Algebra Book. This activity is a combined geometry and algebra activity. It was done at the middle school level, but our boys were ready for it. I reworked the activity that is usually done in 5 sessions so they could do it in 2 sessions. We will do the actual building of Fitz the Pieces and finding it's area and parameter in the session in 2 weeks' time as next week we do not have early release.

I started them with a number a number of puzzles and a number of Triangle pieces that were Large, Medium and Small and asked them to complete the puzzles with these shapes(see figures). I asked them to think and discuss how the sides of these 3 triangles are related to each other, and also to see if they could find the angles for these Triangles.

I left the room and gave them about 7 or 8 minutes, and I could hear that there was a lot of discussion which was great. When I came back, they had done a few of the puzzles and I helped them complete the rest and discussed what elations that they could see. Cyrus was more vocal and active in finding these relations, but Marc is always a bit more quiet but you can see that his computer is working and he was engaged. I helped them discover and say the following:

1. The hypotenuse of each triangle is twice the short leg of that triangle.
2. The hypotenuse of the small triangle is equal to the log leg of the medium triangle
3. The short leg of the large triangle is equal to the long leg of the small triangle.
4. The hypotenuse of the large triangle is equal to 3 short legs of the medium triangle and 2 long legs of the small triangle.
5. 3 short legs of the medium triangle is equal to 2 long legs of the small triangle.
6. The long leg of the large triangle is equal to 3 short legs of the small triangle.

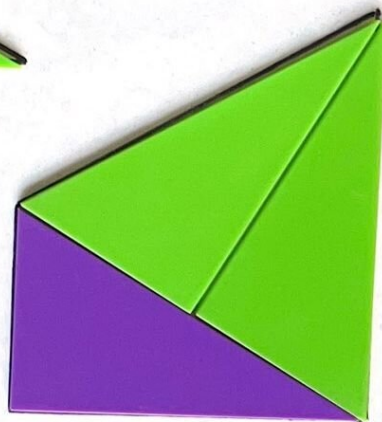
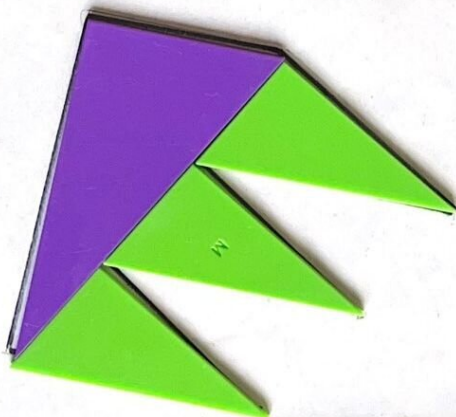
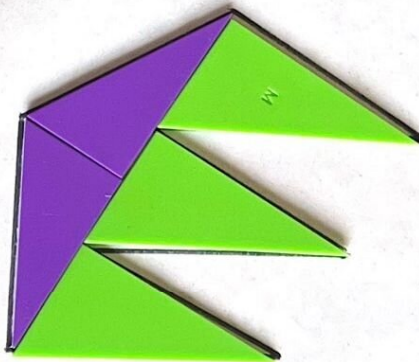
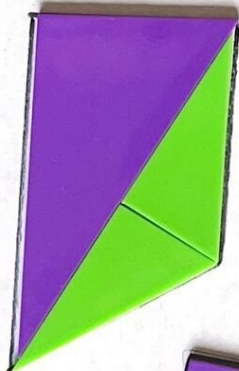
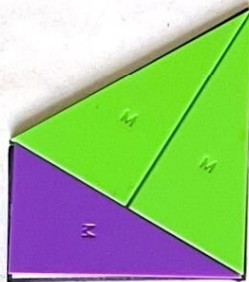
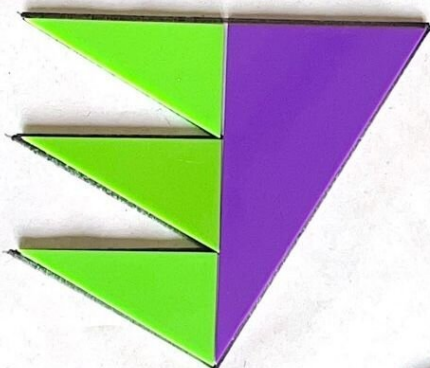
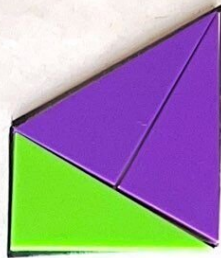
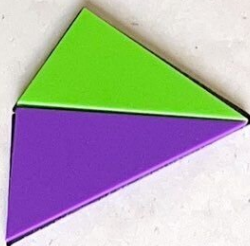
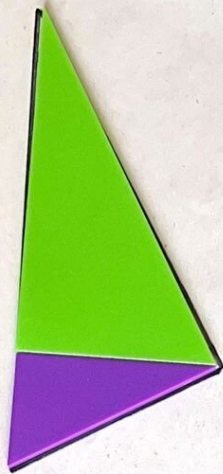
I also had them rearrange the 3 small triangles to see that 3 of the small angles made a right angle so the small triangle was a 30, 60, 90 right triangle. They said that for the medium and the large triangle the same was true. They worked on this together and were completing each other's sentences;=> I also told them since these triangles had corresponding equal angles they were similar. I had them come up with the word similar and that was fun. I also told them that these triangles must have proportional sides and we will try to see this at the end if we have time. Amazingly we got to do this as the session got too long but by the end I had them and they could see that they were doing some cool and advanced new stuff.

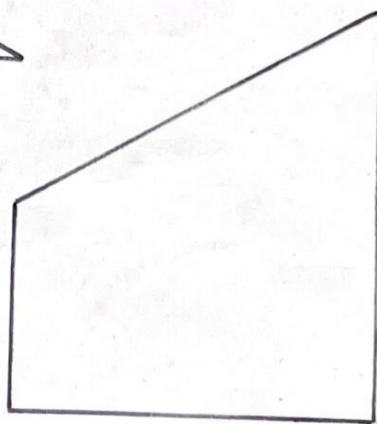
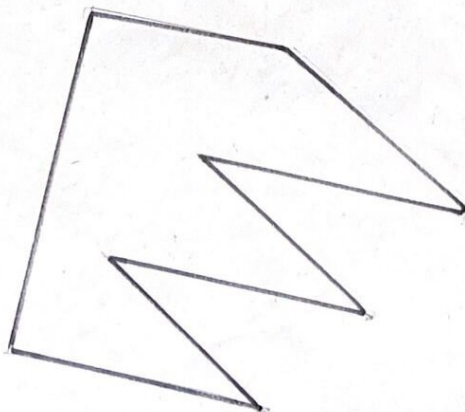
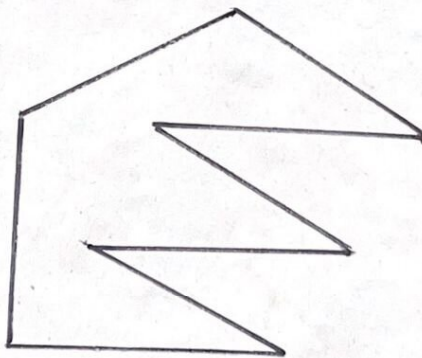
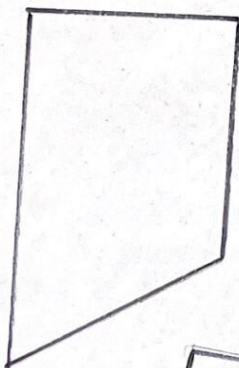
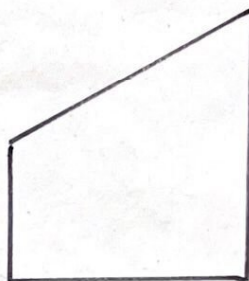
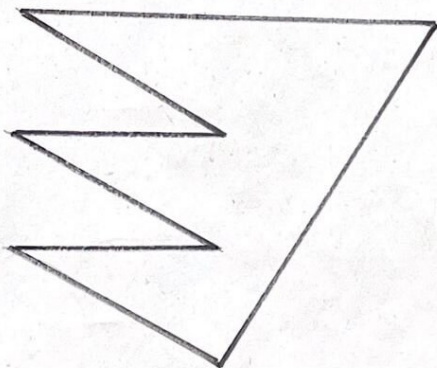
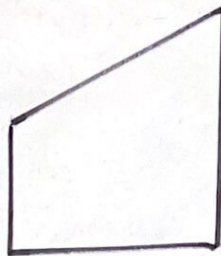
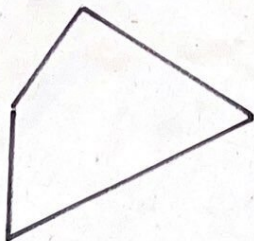
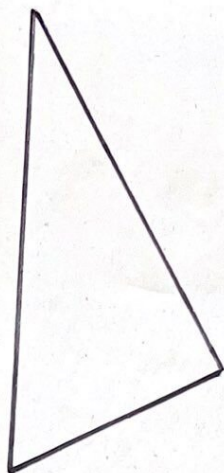
Their next task was to find all the sides of the 3 triangles a, b, c for the small triangle (S), a', b', c' for the medium triangle (M), and a'', b'', c'' for the large triangle (L) (I explained for them that we used prime and double prime notation to mark similar places for shapes in geometry). in terms of the long leg a and the short leg b of the small triangle using the relations that they had found. They still had the puzzles in front of them so they could look at them and figure out the relations again. First, they found that $c=2a$ and then $b'=c=2b$, $b''=a$, $a''=3b$. The rest was a bit more challenging. As they had to compare 2 values, for example $c''=2a$ and $c''=3b'$ so they had to conclude that $3b'=2a$ and $b'=2a/3$ and $c'=4a/3$. They took their time and did a good job.

Next, they had to find the perimeter and area of the 3 triangles. I encouraged them to write down the steps and Cyrus listened to me but Marc as usual just wanted to write the answer. He would come up with the correct values but refused to write down all the steps in good handwriting to get the result. I think his computer works so fast that he does not want to be bothered with the intermediate steps, but it would serve him well to slow down a bit. They were quite engaged and finished the work.

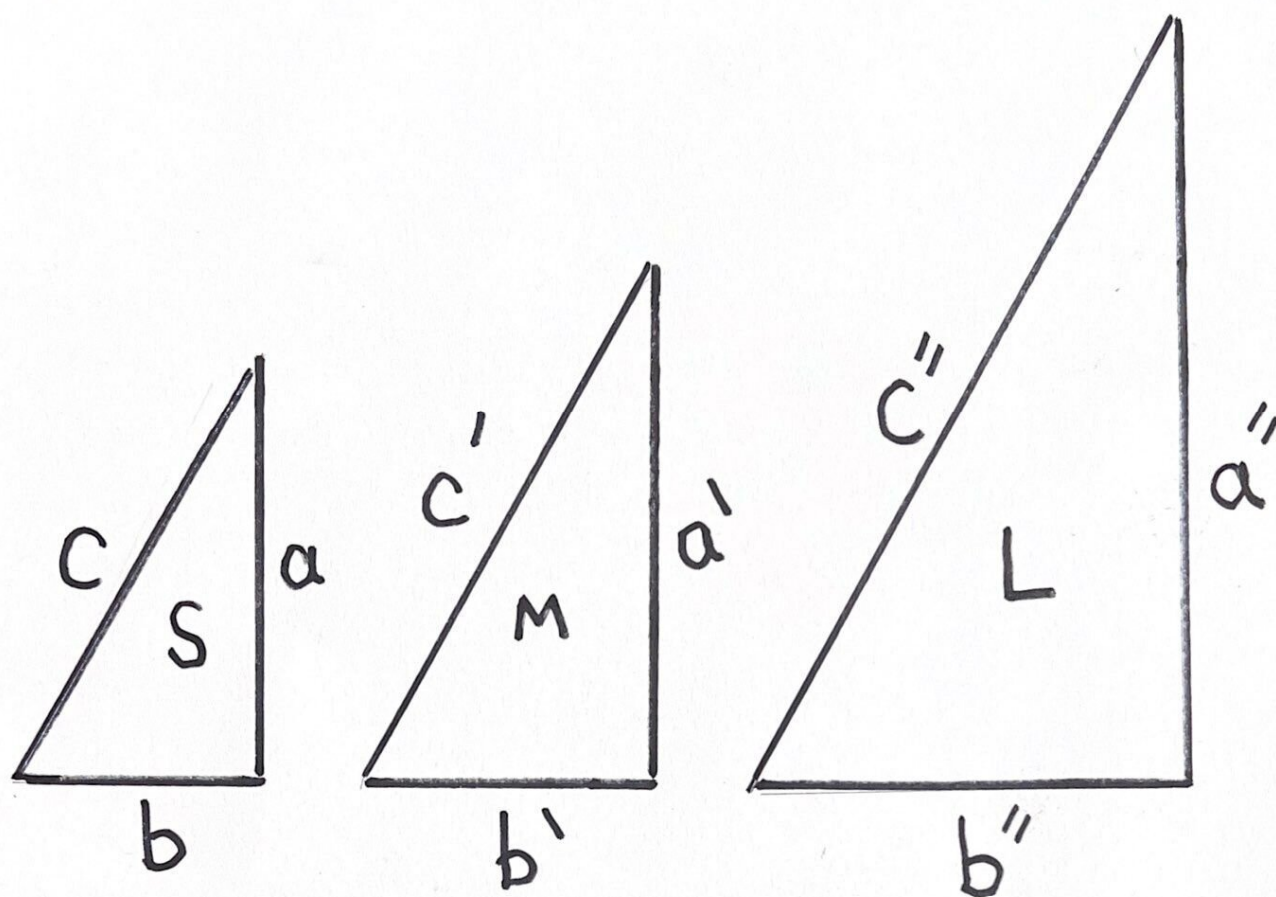
Finally I gave them the worksheet for finding the ratio of the sides of the 3 triangles two by two. Marc complained that he wanted to stop but I managed to encourage him to do the activity. First for Medium to large triangles they found that the ratios were $2/3$ but they needed to relearn how to divide two fractions or a fraction by a number. Next, they did the ratios for small to large triangles and they found that to be $a/3b$ for the long legs and b/a for the other 2 sides. I helped them with using the Pythagorean theorem for the small triangle to find $a^2=3b^2$ and also that if they set $a/3b=b/a$ they would get $a^2=3b^2$. I had to show them how to cross multiply but they were hooked, and both had an aha moment there. Next they did the same for the ratio of the small to medium triangles and they got $a/2b$ for the long leg and $3b/2a$ for the short leg and the hypotenuse. Cyrus immediately said that this was the same as the one that they just did, and Mark agreed. But, I had them write it out and Cyrus even insisted that he wanted to write all the steps. They were quite impressed by this as I was when I did it to prepare for them because I was surprised that the 3 ratios were not the same and delighted when I saw that I could use the Pythagorean theorem to make them equal.

This was one of the best sessions we have had as far as the math content and their full engagement goes. The only other sessions that compare with this one was the ones with the divisibility games last year.





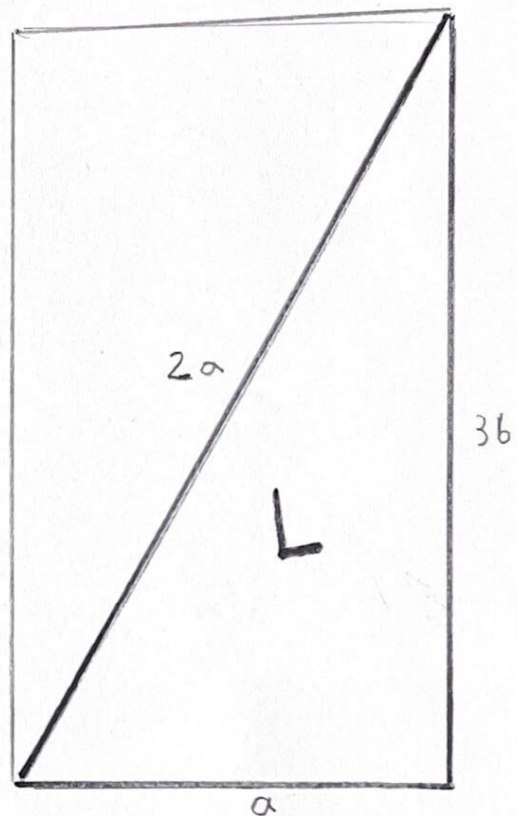
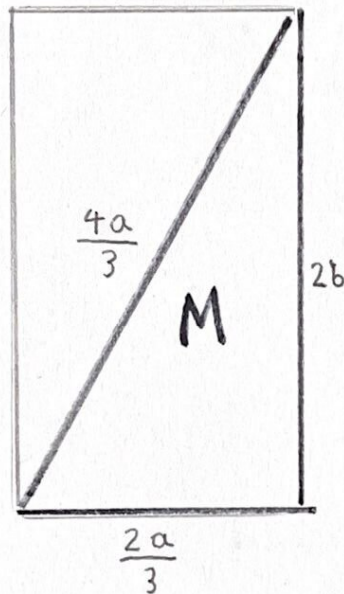
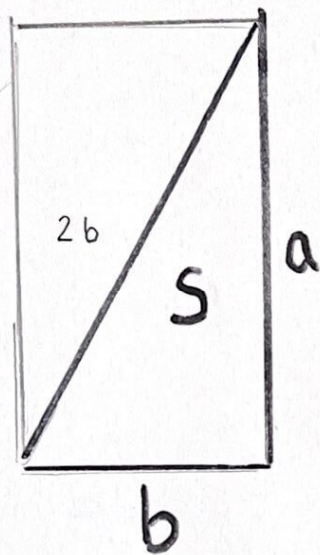
Cyrus



Find all the sides of the 3 triangles in terms of a and b :

	Long Leg	Short Leg	Hypotenuse
S	a	b	$c = 2b$
M	$a' = 2b$	$3b' = 2a$ $b' = \frac{2a}{3}$	$c' = \frac{4a}{3}$
L	$a'' = 3b$	$b'' = a$	$c'' = 2a = 3b'$

Cyrus



1. Mark the sides of the 3 triangles in terms of a and b .
2. Find the perimeter and area of each triangle in terms of a and b

Perimeter	
S	$a + b + 2b = a + 3b$
M	$\frac{2a}{3} + \frac{4a}{3} + 2b = \frac{6a}{3} + 2b = 2a + 2b = (2)(a + b)$
L	$a + 2a + 3b = 3a + 3b = (3)(a + b)$

Area	
S	$\frac{ab}{2}$
M	$\frac{2ab}{3}$
L	$\frac{3ab}{2}$

Worksheet for Side Ratios

Reminders:

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \quad \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

$$\frac{2b}{4a} = \frac{2b}{4a} = \frac{2b}{4a} = \frac{2b}{4a}$$

$$\frac{a}{2b} = \frac{3b}{2a} \Rightarrow a^2 = 6b^2 \quad a^2 = 3b^2 \checkmark$$

$$\frac{a}{3b} = \frac{b}{a} \Rightarrow a^2 = 3b^2 \quad a^2 + b^2 = 4b^2 \quad a^2 = 3b^2$$

$$\frac{a}{2b} = \frac{3b}{2a} \quad 2a^2 = 6b^2 \quad a^2 = 3b^2 \quad \checkmark$$

Find the ratio of the similar sides of the triangles using their values in terms of a and b:

Ratio	Long Leg	Short Leg	Hypotenuse
M/L	$\frac{2b}{3b} = \frac{2}{3}$	$\frac{\frac{2a}{3}}{a} = \frac{2a}{3a} = \frac{2}{3}$	$\frac{\frac{4a}{3}}{2a} = \frac{4a}{6a} = \frac{4}{6} = \frac{2}{3}$
S/L	$\frac{a}{3b} = \frac{b}{a}$	$\frac{b}{a}$	$\frac{8b}{8a} = \frac{b}{a}$
S/M	$\frac{a}{2b} = \frac{3b}{2a}$	$\frac{b}{1} = \frac{3b}{2a}$	$\frac{2b}{1} = \frac{6b}{4a} = \frac{3b}{2a}$

Math Circle 3 08-30-2023 1:55 to 3:15 pm

Cyrus, Marc, and Dar

We continued with Fitz the Pieces activity. First, I showed them Fitz the Pieces that I had made. I also gave them the table that they had made (Cyrus' Copy) that showed all the sides of the 3 triangles and their area and perimeter in terms of **a** and **b** that they had found last time. I also told them that the only rule was to connect the triangles so the sides would fit each other. I briefly reminded them of what they had discovered in the last session in terms of these relations.

The hypotenuse of each triangle is twice the short leg of that triangle.

1. The hypotenuse of the small triangle is equal to the long leg of the medium triangle
2. The short leg of the large triangle is equal to the long leg of the small triangle
3. The hypotenuse of the large triangle is equal to 3 short legs of the medium triangle and 2 long legs of the small triangle.
4. 3 short legs of the medium triangle is equal to 2 long legs of the small triangle.
5. The long leg of the large triangle is equal to 3 short legs of the small triangle.

We had a discussion that they could really make any combination of pieces but if they did not fit exactly calculation of the perimeter would be much harder.

They started and put in a good 20 minutes making their designs. Marc wanted to make a symmetric design and Cyrus did a big bird design.

I helped them put double sided tape on the triangles so that they would not move when they traced them. When I did my monster to test the activity, I did not use the tape and the pieces kept moving. This part took a little bit of time also. I asked them to think about how they would find the area and the perimeter of their design as they were working.

Marc went much faster than Cyrus as usual, but Cyrus did try to keep up. Also, Cyrus paper ripped a bit when we removed one of the Triangles that was taped, but he agreed to fix it and go on (something that he would not have agreed to last year and he would have wanted to start from scratch).

Next, they traced the outline of their design with pencil and removed the triangles one by one and traced them also in pencil. At this point Cyrus said since we have the same number of triangles of each size only one of us must find the area and the other can copy. I told him that that was a great observation, but they had to do it separately and make sure they got to the same area.

They traced all the triangles in the design with ruler and black sharpie and marked them as S, M, and L for small, medium, and large. Then they proceeded to color them with big new sharpies I had gotten for them for this activity and others. Marc was way ahead of Cyrus and colored the pieces so vigorously that we are going to have it on the table for good! This was my fault, but I had not seen the color go through the paper because both Cyrus and I had not gone over the coloring several times that Marc did. I will remember and give them other paper to use under the piece that they are coloring in the future.

Next Marc found the perimeter of his Diamond design and I helped him a bit as they are both not quite sure how numbers and symbols will mix at this point. To his credit he wrote slower and much nicer than before. I also helped him with the area but he quickly said that there were 4 large, 6 medium and 8 large and that way we could multiple the area of each triangle by the number for that size. I helped him go step by step and write each of the steps to get to the answer. This was great as he would have wanted to scible down (the correct answer) before! I had to help him with the simplification of the fractions like $12ab/2$ to realize that he had to treat **a** and **b** just like they were numbers multiplying 12 to get the answer $6ab$.

Cyrus was catching up but he put his coloring on hold to find the perimeter of his Big Bird that was more difficult. So, I also helped him organize the terms and get the final result in terms of a and b . Then he also did the area and compared it with Marc's result. To his credit Cyrus continued and completed the coloring.

It was a very good session and they enjoyed it very much and so did I.

Cyrus

Find all the sides of the 3 triangles in terms of a and b :

	Long Leg	Short Leg	Hypotenuse
S	a	b	$c = 2b$
M	$a' = 2b$	$3b' = 2a$ $b' = \frac{2a}{3}$	$c' = \frac{4a}{3}$
L	$a'' = 3b$	$b'' = a$	$c'' = 2a = 3b$

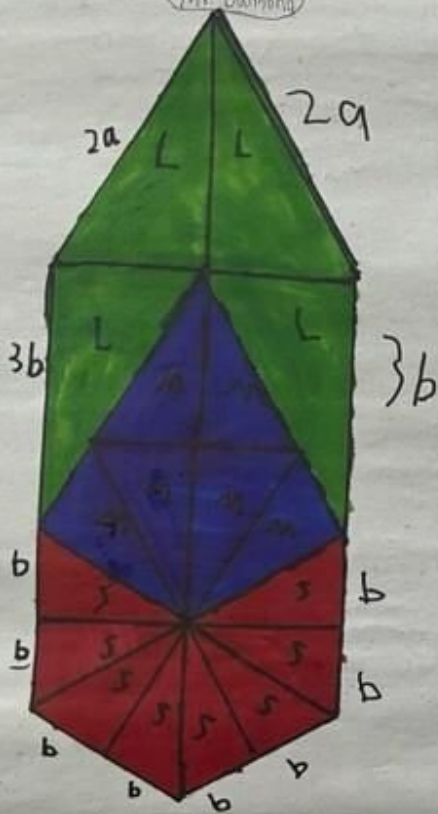
Cyrus

1. Mark the sides of the 3 triangles in terms of a and b .
2. Find the perimeter and area of each triangle in terms of a and b

Perimeter	
S	$a + b + 2b = a + 3b$
M	$\frac{2a}{3} + \frac{4a}{3} + 2b = \frac{6a}{3} + 2b = 2a + 2b = (2)(a+b)$
L	$a + 2a + 3b = 3a + 3b = (3)(a+b)$

Area	
S	$\frac{ab}{2}$
M	$\frac{2ab}{3}$
L	$\frac{3ab}{2}$

Mr. Diamond



$$P = 14b + 4a$$

$$Q = 4 + 6M + 8S$$

$$= \frac{14b}{2} \times 4 = \frac{14ab}{2} = 7ab$$

$$= \frac{24b}{2} \times \frac{2}{3} = \frac{12ab}{3} = 4ab$$

$$= \frac{8b}{2} \times 8 = 4ab$$

$$= 4ab + 4ab + 8ab = 14ab$$

$$Q = 14ab$$

Marc

Big Bird

$$P = 12b + \frac{14a}{3} + 4a$$

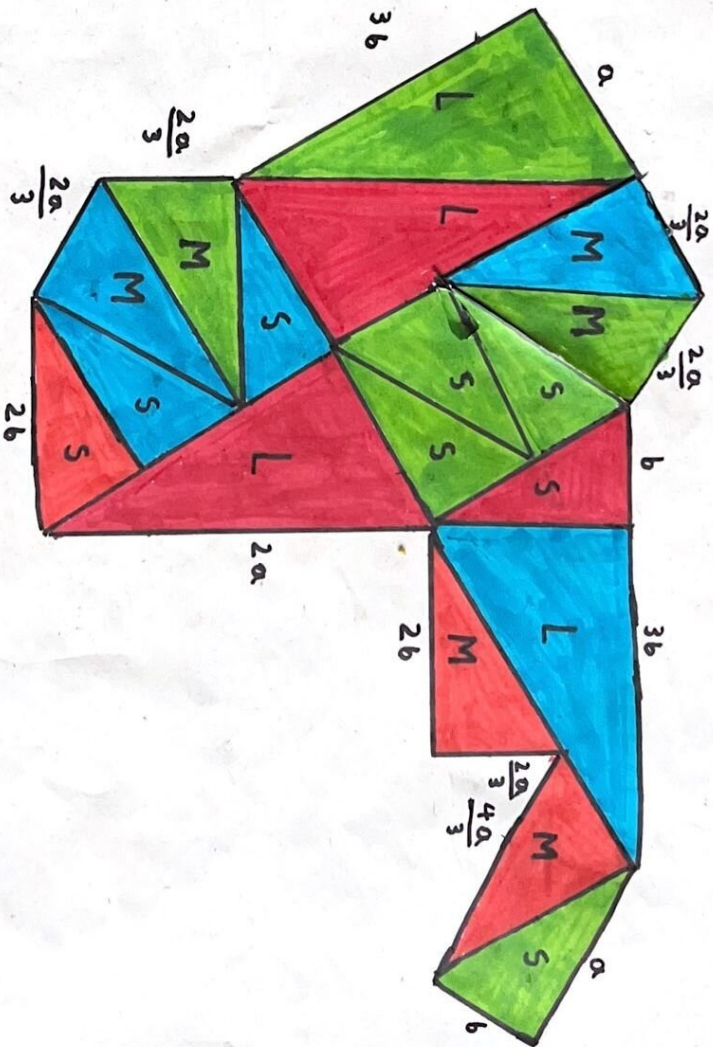
$$P = 12b + \frac{14a}{3} + \frac{12a}{3}$$

$$P = 12b + \frac{26a}{3}$$

$$a = 4L + 6M + 8S$$

$$a = 60b + 4ab + 4ab$$

$$a = 14ab$$



Cyrus

Math Circle 5 11/01/2023 1:10 to 2:20 pm

Cyrus, Marc, and Dar

We continued with another activity from Dr. Super's Triangles Algebra Book, this one was called Hratio F.D. Rectangles and dealt with ratios of the triangles and a number of rectangles to each other.

First, they covered Rectangles A to G with the Small, Medium and Large triangles. While doing this they completed the top part of the first hand out recording how many triangles of what kind covered each rectangle. Some of the rectangles could be covered with different triangles. For example: Rectangle C can be covered with 6 small or 2 large triangles and most importantly Rectangle G can be covered with 8 Large, or 18 Medium or 24 Small Triangles and the relation between all the rectangles and all the Triangles can be derived from this fact.

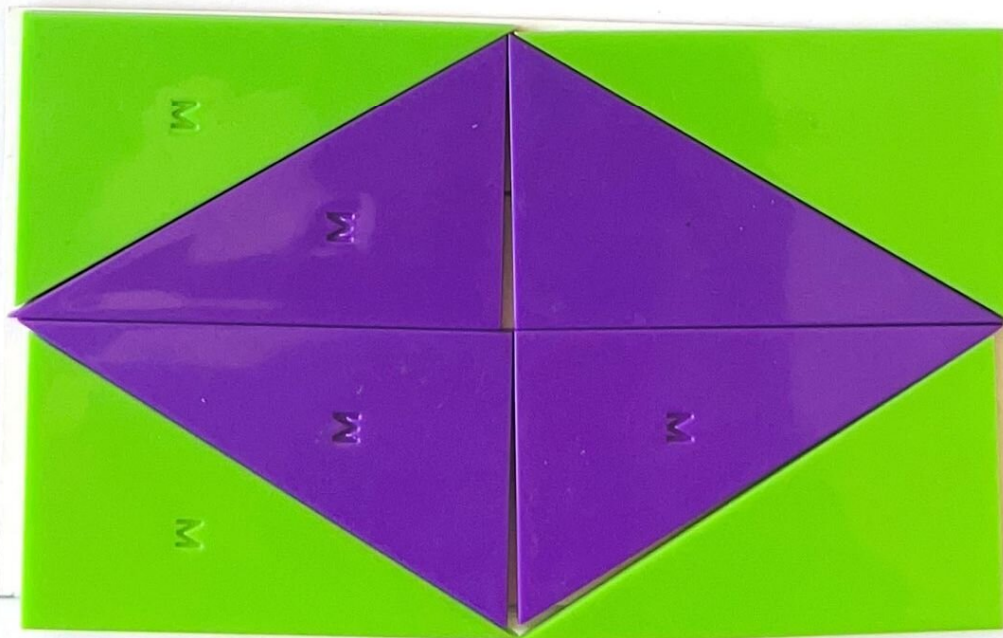
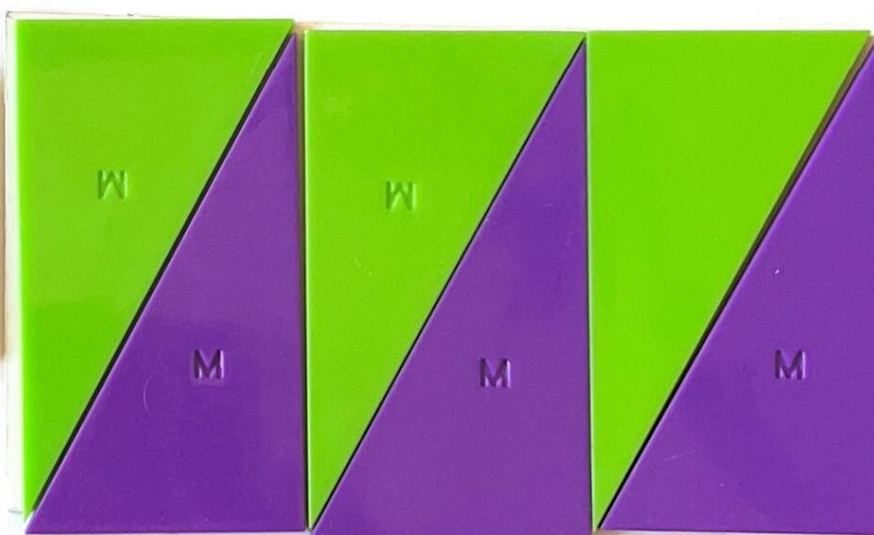
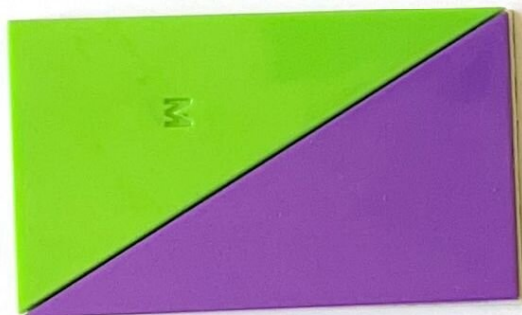
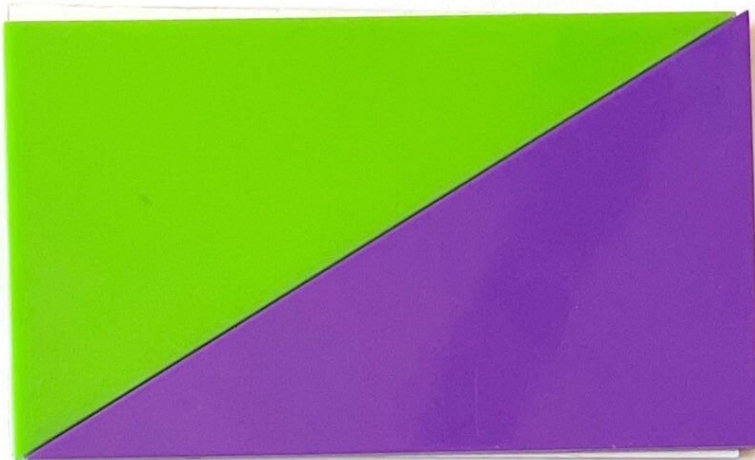
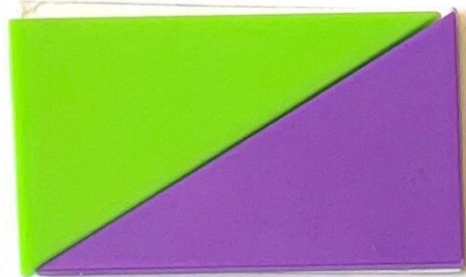
Next, they completed the table of the relation between the Rectangles and the triangles when the ratio was a whole number. For example, $A=2S$ or $G=24s$, $G=18M$ or $G=8L$. Once this was done I suggested that they use the relations found for G to relate S to L, S to M, and M to L. First, they wanted to complete the table without this step but I convinced them that that was the best way to go and they realized that without this step, it was difficult to find many fractional relations. They completed all the relations between S, M and L Triangles using the relations found in Rectangle G and from that completed the fractional relations between the rectangles and the 3 types of Triangles. This took some time but once they got going it speeded up and they enjoyed the process and the fractional reasoning. For example for Rectangle D they reasoned that $B=2M$ since $M=(4/3)S$ then $B=(2 \times 4/3)S=(8/3)S$ and also $B=(2 \times 4/9)L=(8/9)L$.

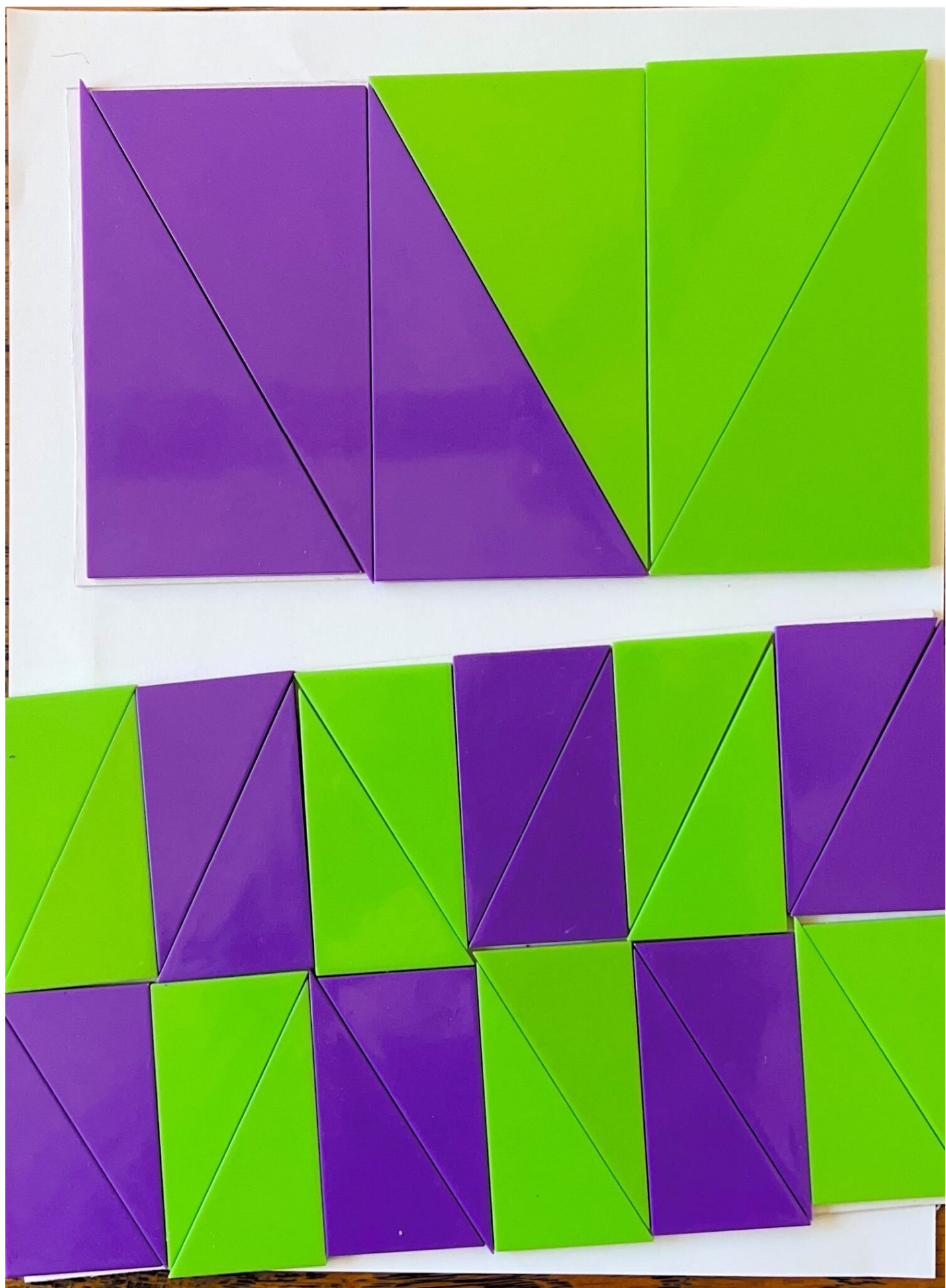
At this point we had a choice but we went ahead to do the other side of the handout, but they did not want to write the sentences relating the triangles to the Rectangles to deduce the ration of the rectangle (Areas) to each other. We discussed the fact that we were really talking algebraically about relation the Areas.

The did this well and discussed it between them and completed the table using the relations that they had found in the previous table. They noticed that there were some repeating patterns. For example, the ratios of the bottom part of the table were less than 1 and the revers of the ratios of the symmetric corresponding cell at the top part of the table that contained ratios greater than 1. They we looking for other patterns but I let them go and play some Ping Pong as they were getting tired.

Roos Crashed the party and wanted to do the activity and started writing the relations on a piece of paper himself.

This was a great activity that they enjoyed. It was designed with a lot more writing for 6th graders to do in a week with an evaluation, but they did it in an hour and ten minutes. I had made some short cuts for them, and Cyrus even objected at the beginning that they did not need what I had done and they could have figured the relations themselves!





A

C

B

D

E

F

G

Horatio F. D. Rectangles

2	Small triangles cover rectangle A
2	Medium triangles cover rectangle B
6	Small triangles cover rectangle C
2	Large triangles cover rectangle C
8	Small triangles cover rectangle D
6	Medium triangles cover rectangle D
8	Medium triangles cover rectangle E
18	Small triangles cover rectangle F
6	Large triangles cover rectangle F
24	Small triangles cover rectangle G
8	Large triangles cover rectangle G
18	Medium triangles cover rectangle G

	S	M	L
A	$A=2S$	$A=\frac{2}{3}M$	$A=\frac{2}{3}L$
B	$B=\frac{8}{3}S$	$B=2M$	$B=\frac{8}{3}L$
C	$C=6S$	$C=\frac{9}{2}M$	$C=2L$
D	$D=8S$	$D=6M$	$D=\frac{8}{3}L$
E	$E=\frac{32}{3}S$	$E=8M$	$E=\frac{32}{9}L$
F	$F=18S$	$F=\frac{27}{2}M$	$F=6L$
G	$G=24S$	$G=18M$	$G=8L$

G	=	24 S
G	=	8 L
G	=	24 S
G	=	18 M
G	=	8 L
G	=	18 M

L	=	3 S
S	=	$\frac{1}{3} L$
M	=	$\frac{4}{3} S$
S	=	$\frac{3}{4} M$
L	=	$\frac{9}{4} M$
M	=	$\frac{4}{9} L$

Horatio F. D. Rectangles

Find and write as many statements as the following Two statements and complete the Ratio Table

2 S Covers 1 A and 6 S Covers 1C so 3A covers 1 C.

2 M covers 1B and 6 M Covers 1D so 3B Covers 1 D

Ratio of the Rectangle Areas

	A	B	C	D	E	F	G
A	1	$\frac{4}{3}$	3	4	$\frac{16}{3}$	9	12
B	$\frac{3}{4}$	1	$\frac{9}{4}$	3	4	$\frac{27}{4}$	9
C	$\frac{1}{3}$	$\frac{4}{9}$	1	$\frac{4}{3}$	$\frac{16}{9}$	3	4
D	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\frac{9}{4}$	3
E	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	$\frac{3}{4}$	1	$\frac{27}{16}$	$\frac{9}{4}$
F	$\frac{1}{9}$	$\frac{4}{27}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{16}{27}$	1	$\frac{4}{3}$
G	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{3}{4}$	1

Horatio F. D. Rectangles

Find and write as many statements as the following Two statements and complete the Ratio Table

2 S Covers 1 A and 6 S Covers 1C so 3A covers 1 C

2 M covers 1B and 6 M Covers 1D so 3B Covers 1 D

Ratio of the Rectangle Areas

	A	B	C	D	E	F	G
A	1	$\frac{4}{3}$	3	4	$\frac{16}{3}$	9	12
B	$\frac{3}{4}$	1	$\frac{9}{4}$	3	4	$\frac{27}{4}$	9
C	$\frac{1}{3}$	$\frac{4}{9}$	1	$\frac{4}{3}$	$\frac{16}{9}$	3	4
D	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\frac{9}{4}$	3
E	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{9}{16}$	$\frac{3}{4}$	1	$\frac{27}{16}$	$\frac{9}{4}$
F	$\frac{1}{9}$	$\frac{4}{27}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{16}{27}$	1	$\frac{8}{3}$
G	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{3}{4}$	1

Horatio F. D. Rectangles

2	Small triangles cover rectangle A
2	Medium triangles cover rectangle B
6	Small triangles cover rectangle C
2	Large triangles cover rectangle C
8	Small triangles cover rectangle D
6	Medium triangles cover rectangle D
8	Medium triangles cover rectangle E
18	Small triangles cover rectangle F
6	Large triangles cover rectangle F
24	Small triangles cover rectangle G
8	Large triangles cover rectangle G
16	Medium triangles cover rectangle G

	S	M	L
A	$A=2S$	$A=\frac{3}{2}M$	$A=\frac{2}{3}L$
B	$B=\frac{8}{3}S$	$B=2M$	$B=\frac{8}{3}L$
C	$C=6S$	$C=\frac{9}{2}M$	$C=2L$
D	$D=8S$	$D=6M$	$\frac{8}{3}L$
E	$E=\frac{32}{3}S$	$E=8M$	$E=\frac{32}{9}L$
F	$F=18S$	$F=\frac{27}{2}M$	$F=6L$
G	$G=24S$	$G=18M$	$G=8L$

G	=	24	S
G	=	8	L
G	=	24	S
G	=	18	M
G	=	8	L
G	=	18	M

L	=	$\frac{1}{3}$	S
S	=	$\frac{1}{3}$	L
M	=	$\frac{4}{3}$	S
S	=	$\frac{4}{3}$	M
L	=	$\frac{6}{8}$	M
M	=	$\frac{9}{8}$	L

Math Circle 6 11/15/2023 1:50 to 3:05 pm

Cyrus, Marc, Rooz and Dar

We continued with another activity from Dr. Super's Triangles Algebra Book, this one was called D. A-Trian. Since Adam could not make it and I was going to do a shortened version of the activity for Adam and Rooz I included Rooz in the Circle with Cyrus and Marc, and he did great and pretty much completed the whole activity with some help on the last work sheet that dealt with the algebraic equation for a line.

I had made outlines for A-Trains, B-Trains and C-Trains on a large newsprint paper and they took turns and cover it with small triangles to make the Trains as I handed out the worksheet based on the lines on the xy coordinates there. First, they did A0, B0 and C0 trains that had no locomotives and 1, 2, or 3 triangles for each car respectively. I did talk a little about the xy-coordinate system but I pretty much let them figure out how they should make the trains and also color them on the worksheet. Next, they took turns and did the same for A-Trains, B-Trains and C-Trains that had 0, 1, 2, or 3 locomotives each, and also colored them on their worksheets. As usual Marc went really fast, and Cyrus was meticulous Rooz also took his time and did all the triangles with different colors. Marc ended up coloring each train with one solid color. Cyrus also did a few trains with a single color eventually, but Rooz persisted and made all his trains with 2 or 3 colors real artistically.

We did talk a lot about why the trains were like that and also about the slope and the intercept and what they meant here even before they started on the last worksheet that covered this topic more in depth. So, they had figured out pretty much that the slope was related to the number of Triangles in a car and the intercept where the line hit the y-axis that was linked to the number of Triangles in the locomotive for that train. So A0, B0 and C0 that had no locomotive all passed through the origin where x and y are both zero; A1, B1, and C1 hit the y-axis at 1; and A2, B2 and C2 hit the y-axis at 2; and A3, B3 and C3 hit the y-axis at 3.

On the next worksheet they actually put in the numbers for the slope and intercept of each train, and we discussed some more the meaning of these terms in terms of a line in general. They also figured out that $y = mx + b$ is the general equation for a line where m is slope and b is the intercept.

Finally, they wrote the equations for 3 lines: $y = 5x + 0$; $y = 0x + 50$ and $y = -1x + 100$. Then they drew these lines on the back of the work sheet on the xy-coordinates. The first one $y = 5x$ was easier to draw by $y = 50$ and $y = -x + 100$ were more difficult

and took some discussion before they all drew them Rooz needed more help here but I think he got the gist of it.

A great session of 1 hour and 15 minutes. But I did have to get Rooz and Cyrus not to engage in verbal ping pong too much;=>. This activity is also done in a week for 6thgrader, of course with a lot more writing and evaluation.

Math Circle 7 11/29/2023 1:55 to 2:55 pm

Cyrus, Marc, and Dar

We continued with another activity with Dr. Super's Triangles that was published under the name Triagram Puzzle. The Book is Exploring Trigonometry with Dr. Super's Trigrams, and it contains 33 challenge activities in 7 Sections. We will cover all the 33 Challenge Activities in 3 sessions. In this Math Circle we covered Challenge Activities 1 to 9 in Sections 1 and 2 and discussed some of the other activities relating to trigonometry in the next session.

Trigram Puzzle is made of 1 Large, 3 Medium and 5 Small Triangles. The original Puzzle fits into a CD case and has a booklet of puzzles similar to the Tangram Puzzle.

The challenges for the first two sections Activity 1 to 9 were like what they had done with Dr. Super's Triangles but now it is assumed that the hypotenuse of the small triangle is 1 unit, and all the sides are computed from that length and the relation between the triangles.

The Pythagorean Theorem was also revisited and used to calculate all the sides of the triangles. These lengths end up being the trig values for 30- and 60-degree angles and we discussed this at the end and will cover it in the next section along with the concept of similarity that they were somewhat familiar with. They had a very good recall of the Pythagorean theorem and were much more comfortable dealing with algebra for the proof. The Visual was also a good starter where they could see that $a^2 + b^2 + 2ab = c^2 + 2ab$ using the small triangles.

Marc got a little tired and wanted to quit to show Cyrus the football project on Scratch and we stopped.

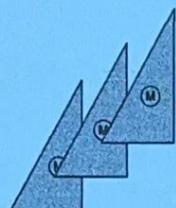
It was a very good session and connected well with what they have been doing. It is a joy for me to do use Dr. Super's Triangle's for these activities as I have not used them for ages. I think they make Algebra and Graphing and Trigonometry fun!

1. Getting Started

What is the Trigram Puzzle?



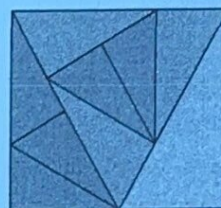
One large triangle



Three medium triangles



Five small triangles



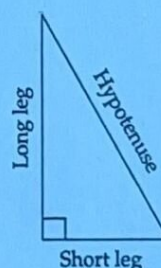
9 Trigram pieces arranged in a rectangle

Pieces of the Trigram Puzzle

Trigram is a set of 9 similar right triangles: 1 large, 3 medium and 5 small triangles as shown above.

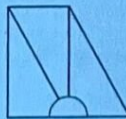
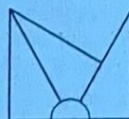
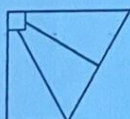
The Names of the Sides of the Trigram Puzzle Pieces

For a right triangle of any size as shown in the example at right, we use the following terminology: The side opposite the right angle is called the hypotenuse; the side opposite the smaller angle is called the short leg; and the side opposite the larger angle is called the long leg.



Sides of a Right Triangle

Challenge Activity 1: Angles in the Trigram Puzzle. Can you find the small angle, large acute angle and the sum of the three angles for each size right triangle in the Trigram Puzzle? (Use the figures below as a hint)



Hints for Challenge Activity 1

Challenge Activity 2: Making Right Triangles. Make as many different right triangles as you can using some or all of the Trigram pieces.

Challenge Activity 3: Trig Animals. You can make many "Trig Animals," and they will help you see relationships between the lengths of the sides of the Trigram Puzzle pieces. The only rule for making a Trig Animal is that triangle sides which are placed alongside each other must add up to the same length. An example would be placing two small triangles with their short legs alongside the hypotenuse of a third small triangle. The two short legs add up to the same length as the hypotenuse, and the overall shape suggests a wolf's head.

Make as many Trig Animals as you can, and state the relationships you discover between the sides of the three triangles.

2. Pythagorean Theorem

One of the most famous theorems in all of mathematics is named after the Greek mathematician Pythagoras, who lived between 600 and 500 B.C. The theorem was known to the Babylonians at least 1,300 years before Pythagoras' time, but no known proof exists before Pythagoras. In this section you will use the Pythagorean Theorem and what we did in the previous activities to find all the side lengths of your Trigram pieces. To find numerical values for the sides we will assume that the hypotenuse of the small triangle is equal to 1 unit.

Challenge Activity 4: Proof of the Theorem Build a square on each side of the small Trigram piece, and use triangle areas to prove $c^2 = a^2 + b^2$. For this proof consider a and b to be the short and long legs of the small triangle. Hint: First, make a square whose sides are $a + b$ on the outside of the square, with an open square inside whose sides are c . Then outline a square whose sides are also $a + b$ on the outside, and which contains the two squares with sides a and b . Compare the area of the shapes that make up these squares.

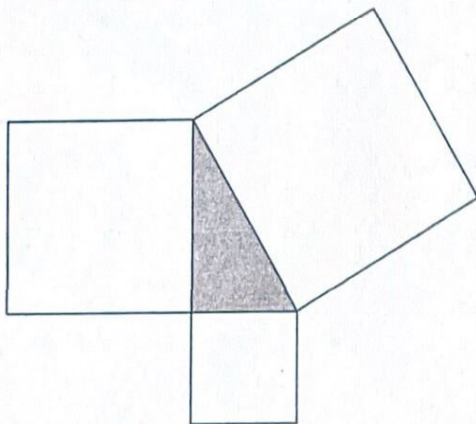
For the next three Activities, assume only that the length of the small triangle's hypotenuse equals one.

Challenge Activity 5: Length Relations I Arrange three small triangles to find the length of the short leg of the small triangle. Use the large triangle and three small triangles to find the length of the long leg of the large triangle.

Challenge Activity 6: Length Relations II Use a large and a small triangle and the Pythagorean Theorem to find the hypotenuse of the large triangle. Use the large triangle and two small triangles to find the long leg of the small triangle.

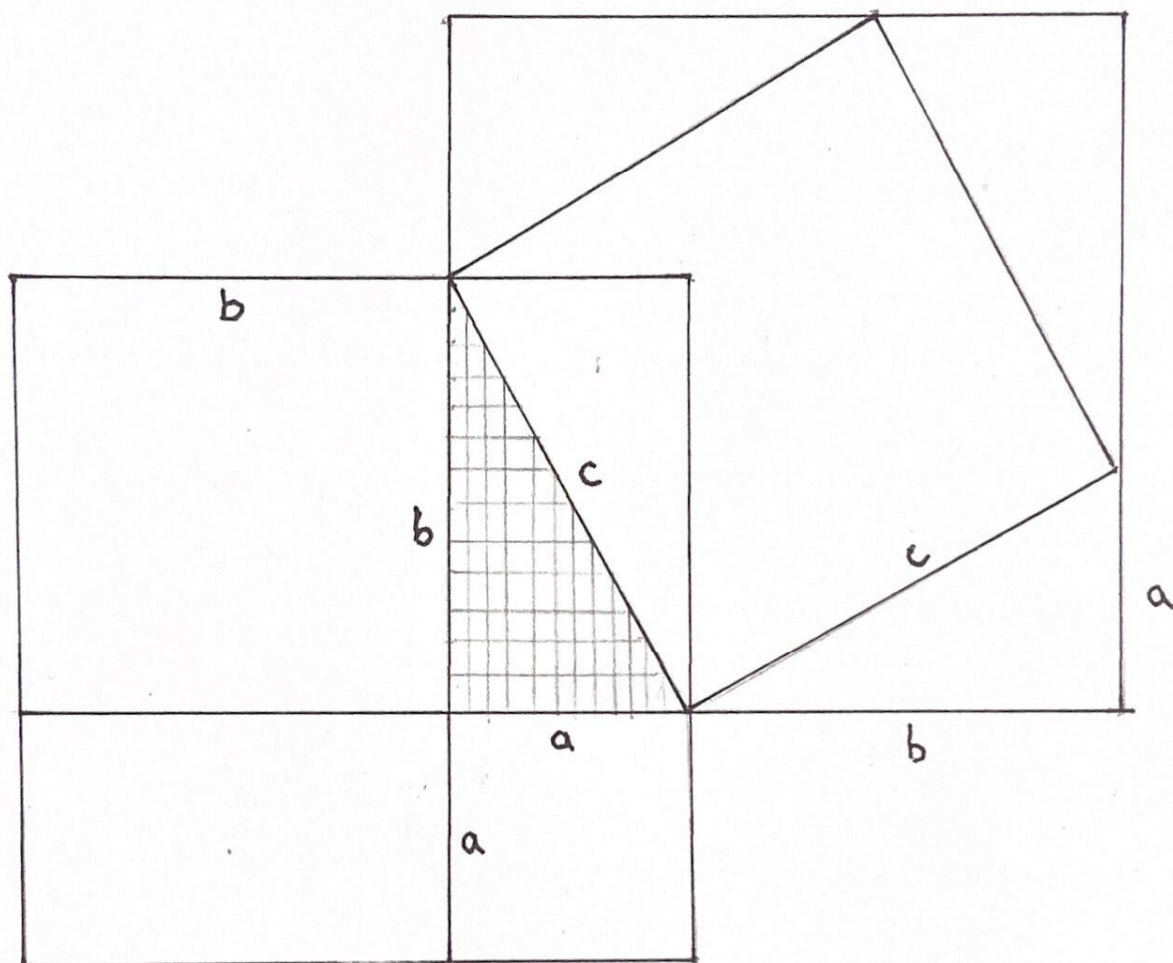
Challenge Activity 7: Length Relations III Use one medium and one small triangle to find the long leg of the medium triangle. Use the large triangle and three medium triangles to find the short leg of the medium triangle. Use three medium triangles to find the hypotenuse of the medium triangle.

Challenge Activity 8: Verifying the Pythagorean Theorem Make a square on each side of the small, medium and large triangles (see example figure below). Find the area of these squares and verify the Pythagorean Theorem for each size triangle.



Verifying the Pythagorean Theorem

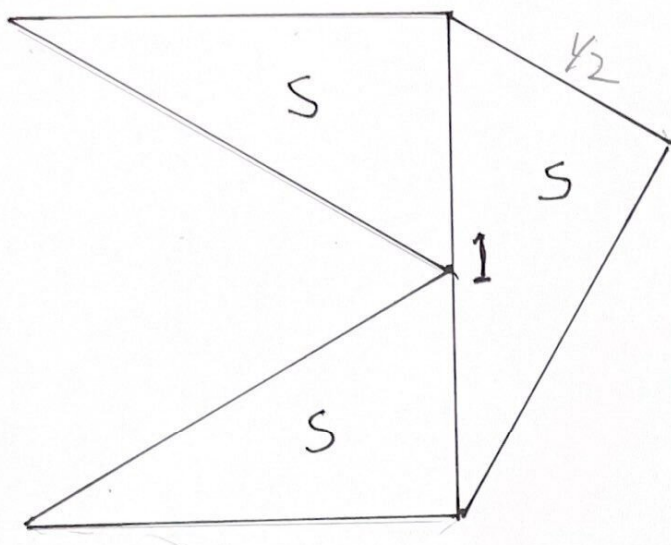
Activity 4: Proof of the Pythagorean Theorem



$$a^2 + b^2 = c^2$$

$$c^2 + 2ab = (a+b)^2$$

$$a^2 + b^2 + 2ab = (a+b)^2 \quad \text{QED}$$

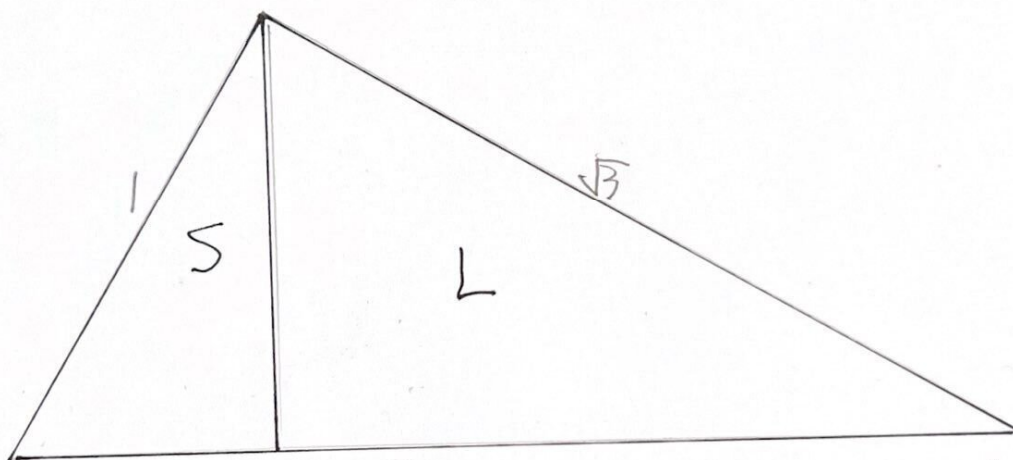


Activity 5:

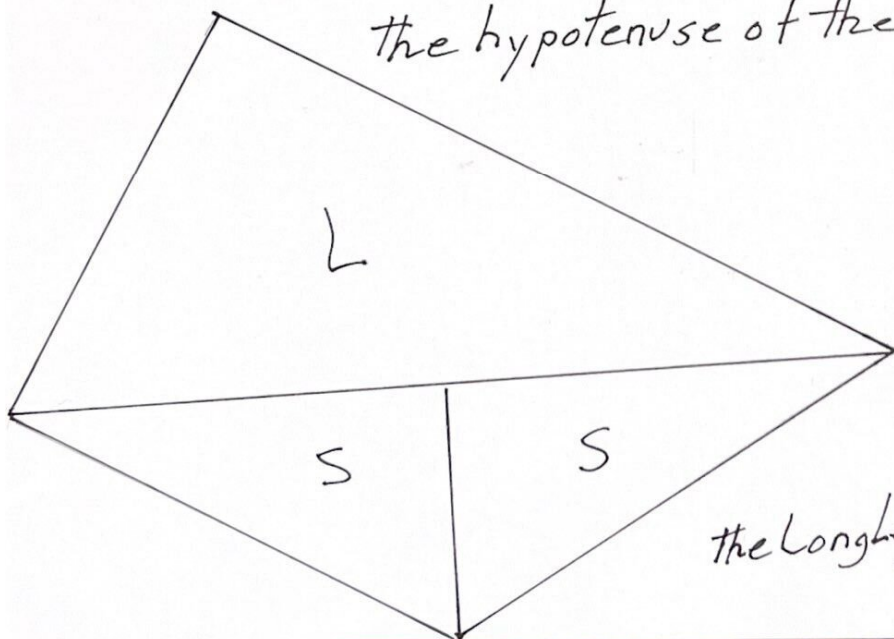
$$C = 1$$

the short leg of small triangle =
 $a =$

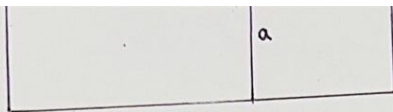
Activity 6:



the hypotenuse of the large triangle = $\sqrt{3}$
 $C' =$



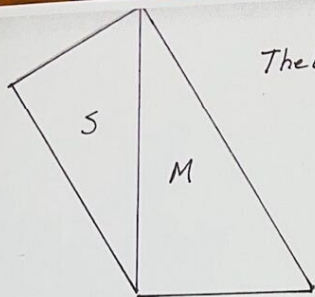
the Long leg of the small triangle =
 $b = \frac{\sqrt{3}}{2}$



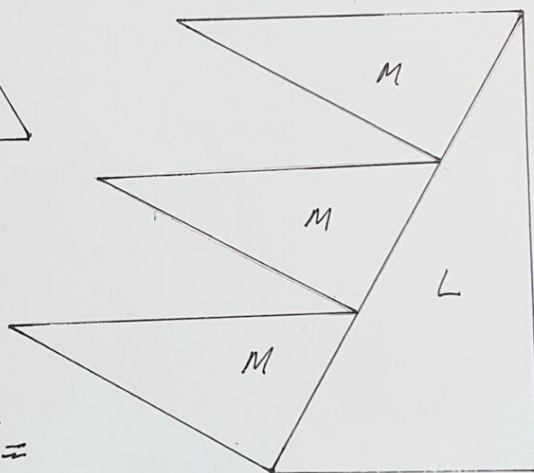
$$a^2 + b^2 = c^2$$

$$c^2 + 2ab = (a+b)^2$$

$$a^2 + b^2 + 2ab = (a+b)^2 \text{ QED}$$



Activity 1:
The Long Leg of the medium triangle =
 $b' =$



the short Leg of the
medium triangle =
 $a' =$

Activity 8: Verify the Pythagorean Theorem for S, M, L

$$a = \frac{1}{2}$$

$$b = \frac{\sqrt{3}}{2}$$

$$c = 1$$

$$a' = \frac{\sqrt{3}}{3}$$

$$b' = 1$$

$$c' = \frac{2\sqrt{3}}{3}$$

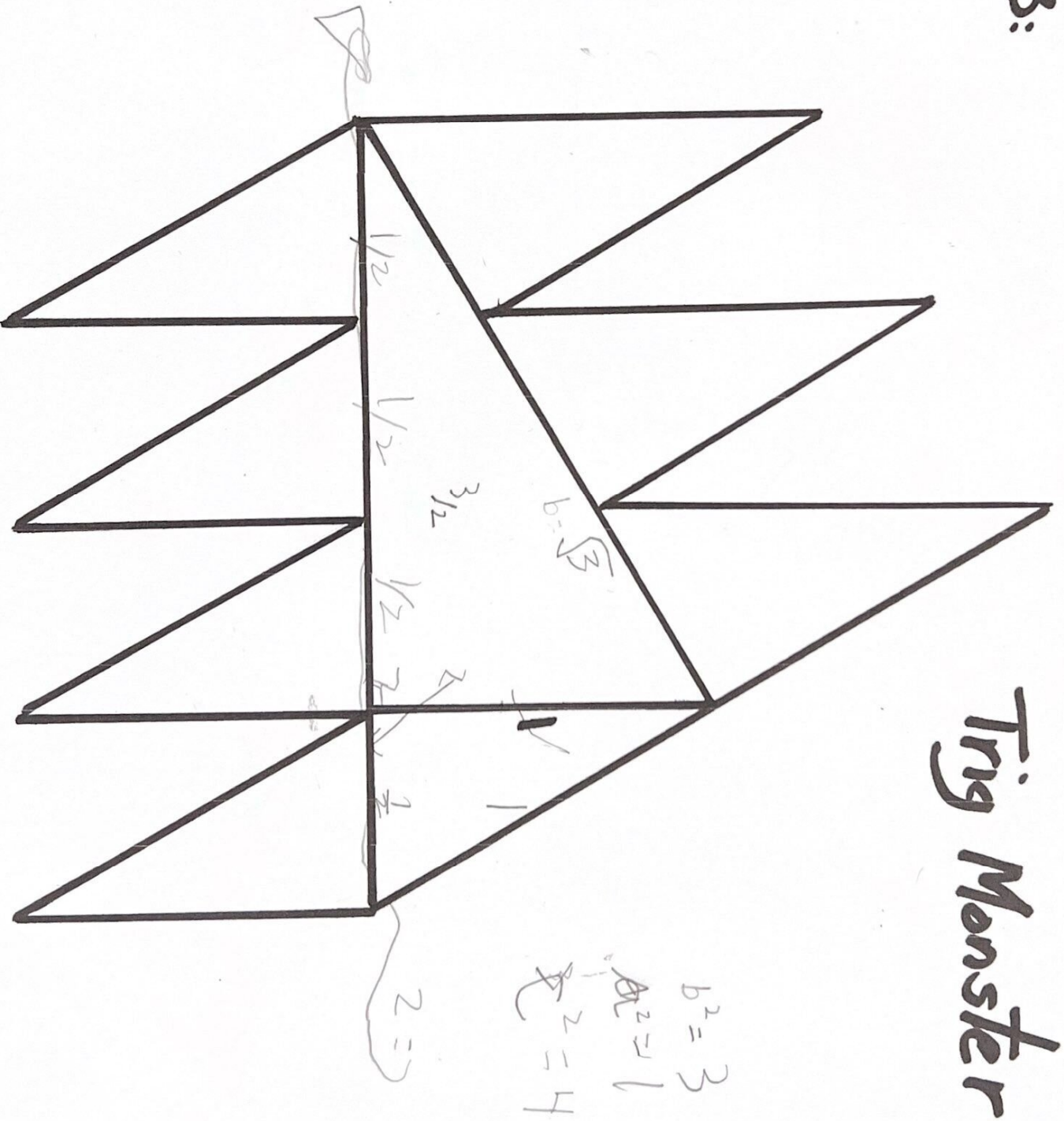
$$a'' = \frac{\sqrt{3}}{2}$$

$$b'' = 3/2$$

$$c'' = \sqrt{3}$$

Activity 3:

Trig Monster



Math Circle 7 11/29/2023 1:55 to 2:55 pm

Cyrus, Marc, and Dar

We continued with the preparatory activities from the book Exploring Trigonometry with Dr. Super's Trigrams, we did some preparatory work with positive and negative powers and square roots so we did not go as fast as I thought we would. We almost finished section 3 that is the start of the real Trig work.

I first showed them a visual demonstration for why any number to the power 0 is 1 and what is a negative power (see figure). The figure is for powers of 2 but we discussed that this would work for any number if instead of dividing by 2 with divide by 3 or 4 etc. and they were comfortable with it.

I also reviewed a way to divide fractions as they had to do a lot that in this session.

Next, they did a work sheet I prepared for them to review positive and negative powers and roots and how the multiplicative rule for them worked. They were very fast doing this, and it was a good review.

Next, they found the sides of their 3 triangles and marked them on their worksheet. They did complain that they had done it last time, but it did take them a bit to recall it all. They went back to what they did, and Cyrus even used the triangles to verify some of the lengths. Next, they verified the Pythagorean Theorem for each triangle.

Now they were ready to check that the 3 triangles were similar by comparing the side lengths. This took a lot of time as there were a lot of fractional divisions with square roots. As usual Cyrus took his time and Marc went fast and got sloppy even though he was getting the right answers. I encouraged Cyrus to go a bit faster and Marc to slow down and write slower. There was a lot of time spent simplifying the terms with the square root of 3 in the denominator that they had practiced before at the start of the session and by the end they were both more comfortable with it. But probably more practice is needed with this. We were running out of time by the time they started to find the $\sin 30$ and see that it is the $\cos 60$ and also the $\sin 60$ and $\cos 30$ but we discussed that these were complementary angles and this will happen with all complementary angles so for example for 45 degree angle $\sin$ and $\cos$ will be the same. We will cover this in much more detail and on the unit circle in the next session.

It was a very good session and I think now they are ready to understand what the trig functions are and even do some simple identities with them.



$$8 = 2^3$$



$$4 = 2^2$$



$$2 = 2^1$$



$$1 = 2^0$$

$$1 = 2^0$$

$$\frac{1}{2} = 2^{-1}$$

$$\frac{1}{4} = 2^{-2}$$

$$\frac{1}{8} = 2^{-3}$$

$$\left(\frac{2}{3} \right)_{\frac{5}{7}} = \frac{2 \times 7}{3 \times 5} = \frac{14}{15}$$

Name: M. N. D.

$$a^n = \underbrace{a \times a \times \dots \times a}_n$$

$$10^4 = \frac{10,000}{1}$$

$$a^{-n} = \frac{1}{a^n}$$

$$2^{-4} = \frac{1}{16}$$

$$a^n \times a^m = \underbrace{a \times a \times \dots \times a}_n \times \underbrace{a \times a \times \dots \times a}_m = a^{n+m}$$

$$10^2 \times 10^3 = 10^5$$

$$2^3 \times 2^{-3} = 2^0$$

$$\sqrt{a} = a^{\frac{1}{2}}$$

$$\sqrt{a^2} = a$$

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{a} \times \sqrt{a} = a$$

$$\sqrt{a^4} = a^2$$

Never keep the squareroot
in the denominator

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\frac{3}{\sqrt{3}} = \frac{\sqrt{3} \cancel{\sqrt{3}}}{\cancel{\sqrt{3}}} = \sqrt{3}$$

Name: Cytus

$$a^n = \underbrace{a \times a \times \dots \times a}_n$$

$$10^4 = \underline{10,000}$$

$$a^{-n} = \frac{1}{a^n}$$

$$2^{-4} = \underline{\frac{1}{16}}$$

$$a^n \times a^m = \underbrace{a \times a \times \dots \times a}_n \times \underbrace{a \times a \times \dots \times a}_m = a^{n+m}$$

$$10^2 \times 10^3 = \underline{10^5}$$

$$2^3 \times 2^{-3} = \underline{2^0}$$

$$\sqrt{a} = a^{\frac{1}{2}}$$

$$\sqrt{a^2} = \underline{a}$$

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{a} \times \sqrt{a} = \underline{a}$$

$$\sqrt{a^4} = \underline{a^2}$$

Never keep the squareroot
in the denominator

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{\sqrt{3} \times 1}{\sqrt{3} \times \sqrt{3}} = \underline{\frac{\sqrt{3}}{3}}$$

$$\frac{\sqrt{3} \times 3}{\sqrt{3} \times \sqrt{3}} = \underline{\frac{\cancel{\sqrt{3}} \times 3}{\cancel{\sqrt{3}}}} = \underline{3}$$

$$a^2 + b^2 = c^2$$

$$\frac{\sqrt{1}}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

$$1 + \frac{1}{3} = \frac{4}{3}$$

$$\frac{9}{4} + \frac{3}{4} = \}$$

$$a^2 + b^2 = c^2$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

$$\frac{1}{3} + 1 = \frac{4}{3}$$

$$\frac{9}{4} + \frac{3}{4} =$$

$$a^2 + b^2 = c^2$$

$$\frac{1}{4} + \frac{3}{4} = 1$$

$$\frac{1}{3} + 1 = \frac{4}{3}$$

$$\frac{9}{4} + \frac{3}{4} = 3$$

$$\frac{\frac{1}{3}}{\frac{2}{2}} = \frac{\frac{1}{3}}{\frac{3}{2}} = \frac{1 \times 2}{1 \times 3} = \frac{2}{3}$$

$$\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{2}} = \frac{\cancel{\sqrt{3}}}{\sqrt{3} \times 3} = \frac{2}{3}$$

$$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{3}} = \frac{1 \times 3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\sqrt{3}\cancel{3}} = \frac{\sqrt{3}}{2}$$

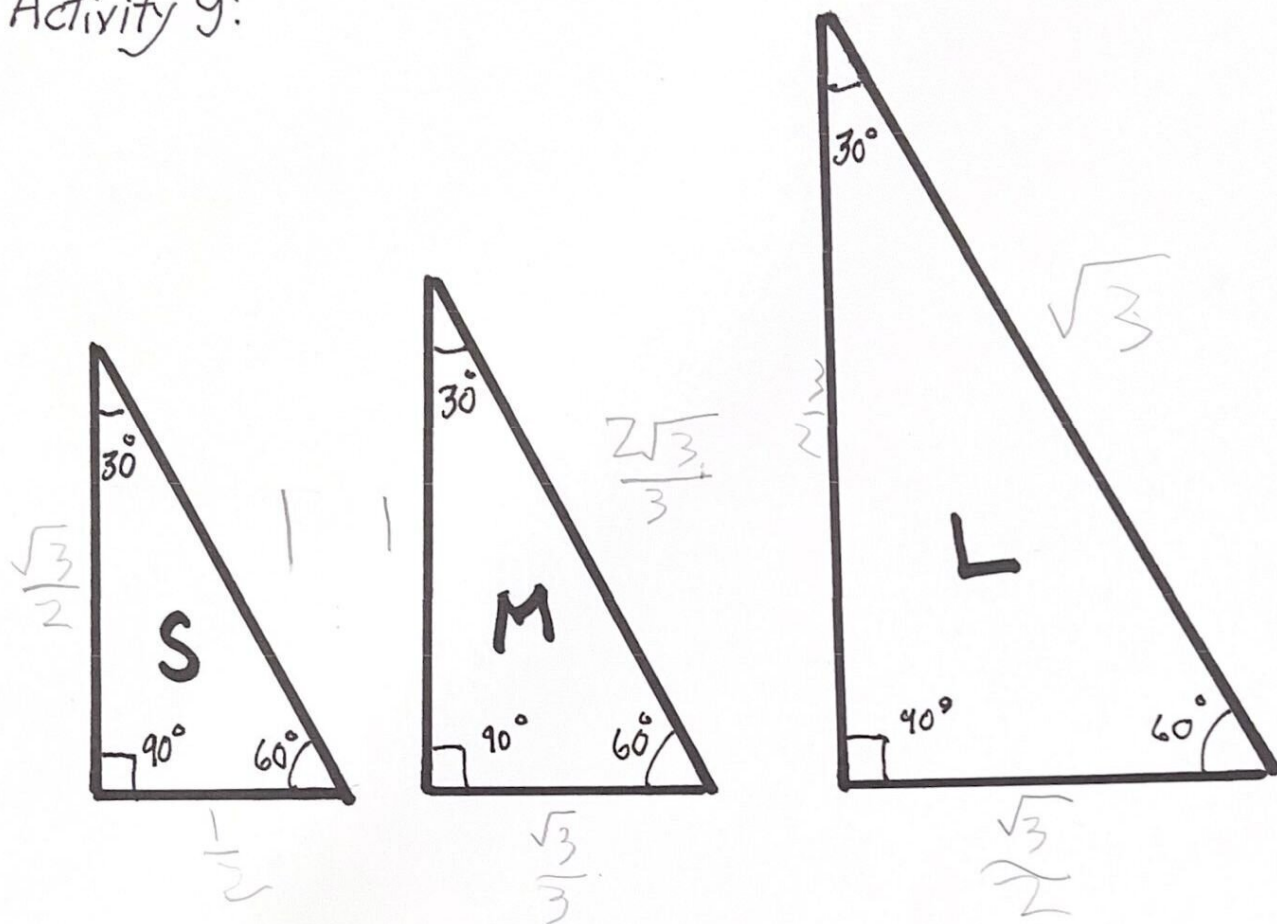
$$\frac{1}{\frac{3}{2}} = \frac{\frac{1}{1}}{\frac{3}{2}} = \frac{1 \times 2}{1 \times 3} = \frac{2}{3}$$

$$\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3} \times 2}{3 \times 3}}{\frac{\sqrt{3} \times 3}{3 \times 3}} = \frac{2\sqrt{3}}{3\sqrt{3}} = \frac{2}{3}$$

$$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{3}} = \frac{\frac{1 \times 3}{2 \times \sqrt{3}}}{\frac{2 \times \sqrt{3}}{2 \times \sqrt{3}}} = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$$

Worksheet for Sides and Side Ratios for Trig

Activity 9:

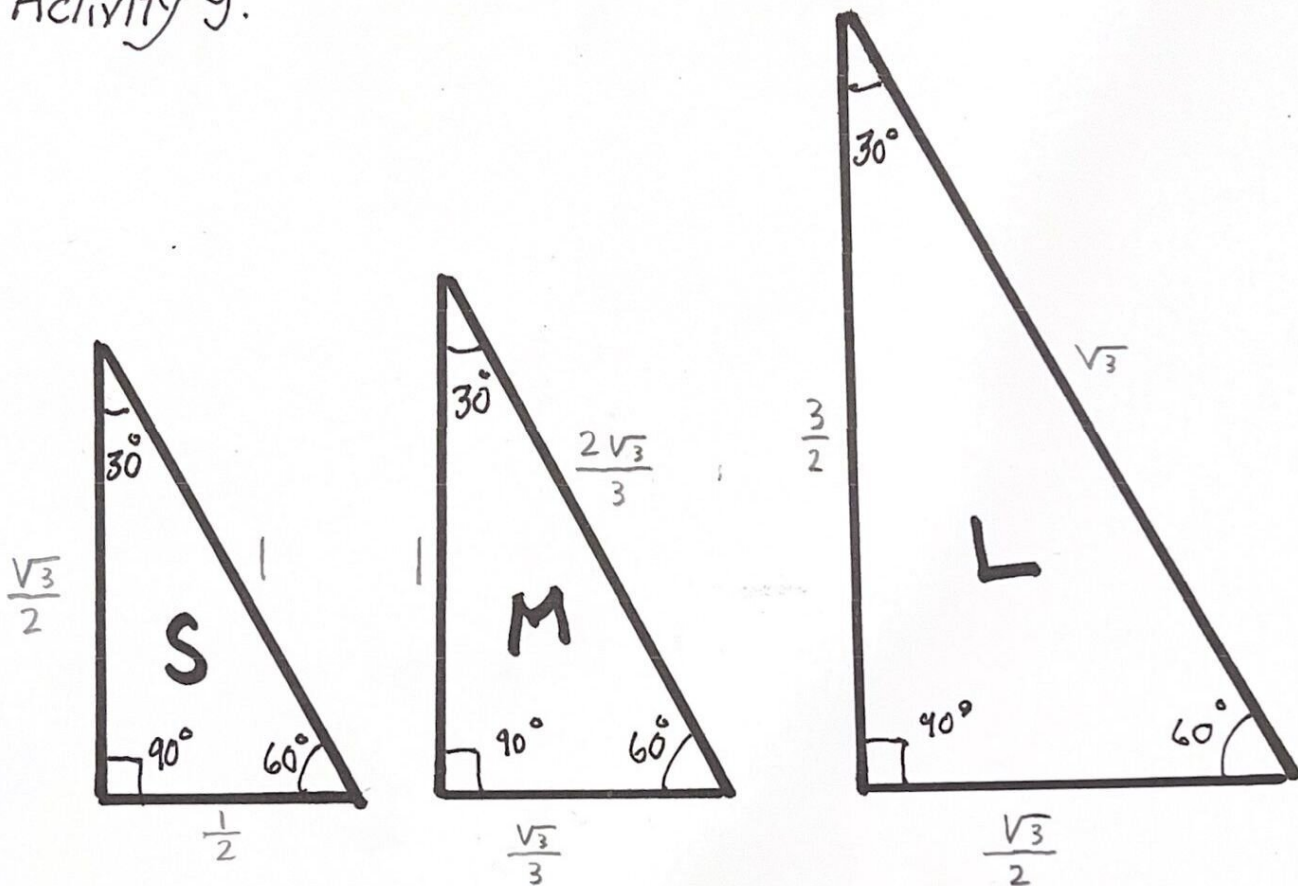


Find the ratio of the similar sides of the 3 triangles

Ratio	Long Leg	Short Leg	Hypotenuse
M/L	$\frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$	$\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{2}} = \frac{2\sqrt{3}}{3\sqrt{3}} = \frac{2}{3}$	$\frac{\frac{2\sqrt{3}}{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3\sqrt{3}} = \frac{2}{3}$
S/L	$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2\sqrt{3}}{1} = \frac{\sqrt{3}}{1}$	$\frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{1} = \frac{1}{1}$	$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
S/M	$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{3}} = \frac{3\sqrt{3}}{2}$	$\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$	$\frac{1}{\frac{2\sqrt{3}}{3}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$

Worksheet for Sides and Side Ratios for Trig

Activity 9:



Find the ratio of the similar sides of the 3 triangles

Ratio	Long Leg	Short Leg	Hypotenuse
M/L	$\frac{1}{\frac{3}{2}} = \frac{1}{\frac{3}{2}} = \frac{1 \times 2}{3 \times 1} = \frac{2}{3}$	$\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{2}} = \frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{2}} = \frac{2 \times \sqrt{3}}{3 \times \sqrt{3}} = \frac{2}{3}$	$\frac{\frac{2\sqrt{3}}{3}}{\sqrt{3}} = \frac{\frac{2\sqrt{3}}{3}}{\frac{\sqrt{3}}{1}} = \frac{1 \times 2\sqrt{3}}{3 \times \sqrt{3}} = \frac{2}{3}$
S/L	$\frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{2 \times \sqrt{3}}{2 \times 3} = \frac{\sqrt{3}}{3}$	$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1 \times 2}{2 \times \sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
S/M	$\frac{\frac{\sqrt{3}}{2}}{1} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3} \times 2}{1 \times 2} = \frac{\sqrt{3}}{2}$	$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{3}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{3}} = \frac{1 \times 3}{2 \times \sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$	$\frac{\frac{1}{\sqrt{3}}}{\frac{2\sqrt{3}}{3}} = \frac{\frac{1}{\sqrt{3}}}{\frac{2\sqrt{3}}{3}} = \frac{3}{1 \times 2\sqrt{3}} = \frac{\sqrt{3}}{2}$

sinus (sin), cosinus (cos), tangant (tan) and cotangant (cotan) of
the 30° Angle and 60° Angle in a 30-60 Right Triangle

30° Angle	
	Ratio of Opposite side to Hypothenuse sin(30)
S	$\frac{1}{2}$
M	$\frac{1}{2}$
L	$\frac{1}{2}$
	Ratio of Adjacent side to Hypothenuse Cos(30)
S	$\frac{\sqrt{3}}{2}$
M	$\frac{\sqrt{3}}{2}$
L	$\frac{\sqrt{3}}{2}$
	Ratio of Opposite side to Adjacent Side tan(30)
S	
M	
L	
	Ratio of Adjacent side to Opposite Side cotan(30)
S	
M	
L	

60° Angle	
	Ratio of Adjacent side to Hypothenuse Cos(60)
S	$\frac{1}{2}$
M	$\frac{1}{2}$
L	$\frac{1}{2}$
	Ratio of Opposite side to Hypothenuse sin(60)
S	$\frac{\sqrt{3}}{2}$
M	$\frac{\sqrt{3}}{2}$
L	$\frac{\sqrt{3}}{2}$
	Ratio of Adjacent side to Opposite Side cotan(60)
S	
M	
L	
	Ratio of Opposite side to Adjacent Side tan(60)
S	
M	
L	

sinus (sin), cosinus (cos), tangant (tan) and cotangant (cotan) of
the 30° Angle and 60° Angle in a 30-60 Right Triangle

30° Angle	
	Ratio of Opposite side to Hypothenuse sin(30)
S	$\frac{1}{2}$
M	$\frac{1}{2}$
L	$\frac{1}{2}$
	Ratio of Adjacent side to Hypothenuse Cos(30)
S	$\frac{\sqrt{3}}{2}$
M	$\frac{\sqrt{3}}{2}$
L	$\frac{\sqrt{3}}{2}$
	Ratio of Opposite side to Adjacent Side tan(30)
S	
M	
L	
	Ratio of Adjacent side to Opposite Side cotan(30)
S	
M	
L	

60° Angle	
	Ratio of Adjacent side to Hypothenuse Cos(60)
S	$\frac{1}{2}$
M	$\frac{1}{2}$
L	$\frac{1}{2}$
	Ratio of Opposite side to Hypothenuse sin(60)
S	
M	
L	
	Ratio of Adjacent side to Opposite Side cotan(60)
S	
M	
L	
	Ratio of Opposite side to Adjacent Side tan(60)
S	
M	
L	

Math Circle 9 12/13/2023 1:55 to 2:55 pm

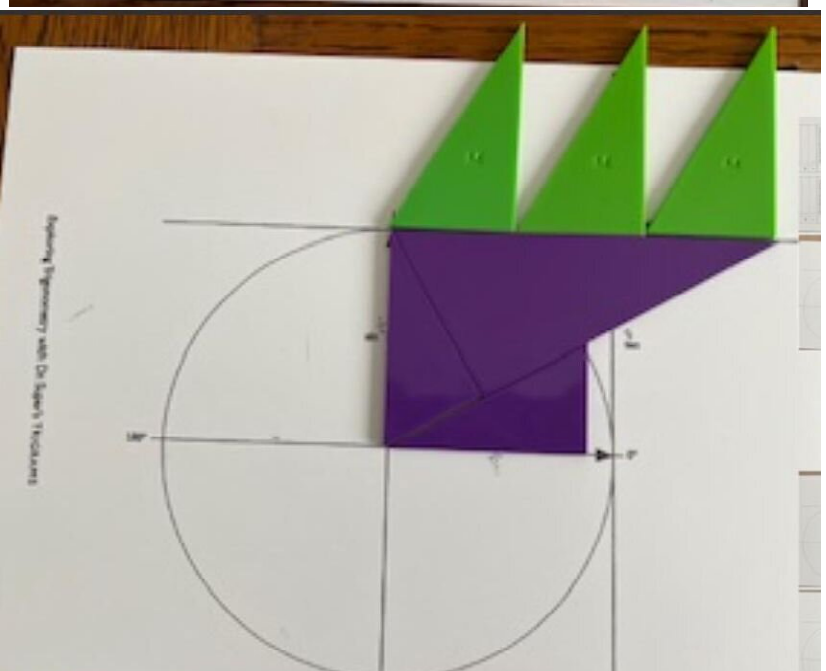
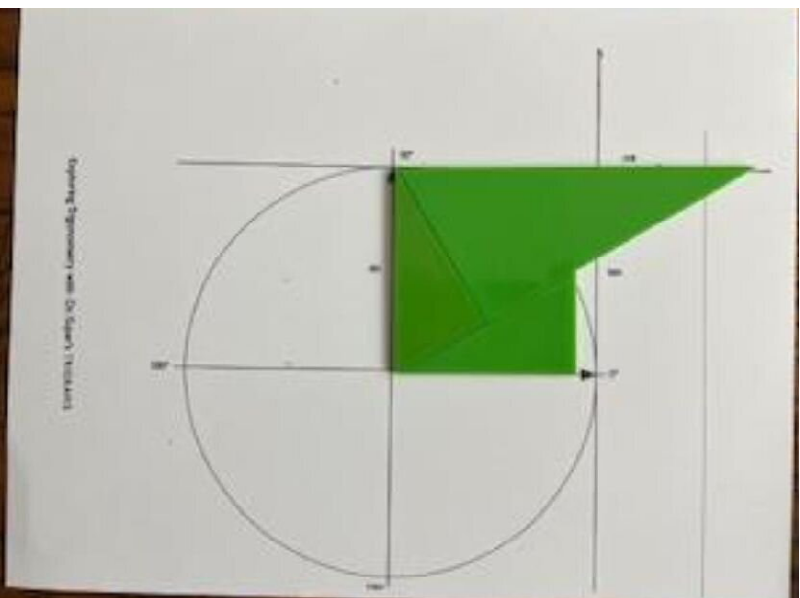
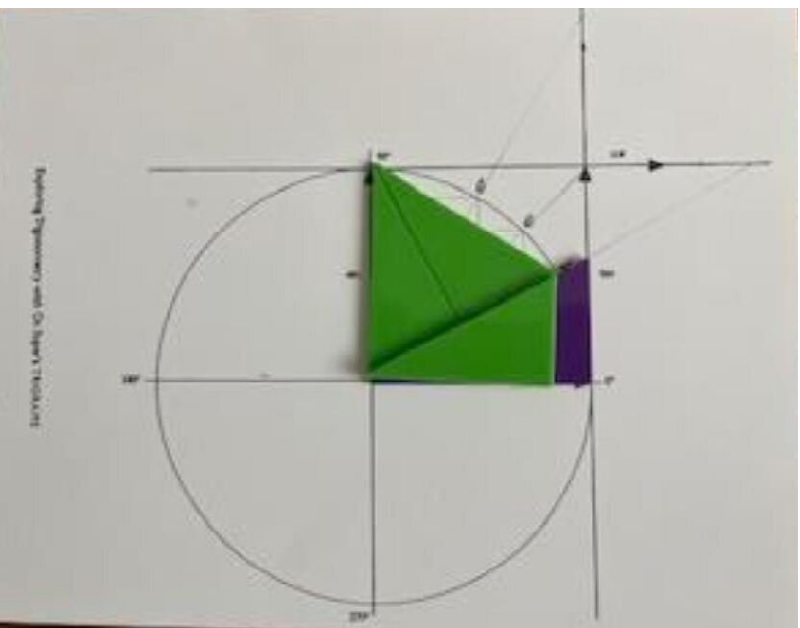
Cyrus, Marc, and Dar

We continued with the activities from the book Exploring Trigonometry with Dr. Super's Trigrams. Cyrus and Marc finished section 3 that is the start of the real Trig work and completed the table that has the sin, cos, tan, and cotan for 30- and 60-degree angles using their triangles. They also saw that for complementary angles like 30 and 60 the sin of 1 is equal to the cos of the other angle and tan is equal to the cotan of the other angle that we had discussed the last time. We also did the activities in Section 4 where the trig functions are connected to their definition on the a unit circle.

We discussed the trig circle and talked about the cos, sin, tan and cotan axis and then they proceeded to recalculate the values for the 30 degree angle using their triangles and the trig circle.

First by placing the small triangle they verified that the sign30 is $\frac{1}{2}$ then using the large triangle and small one they found that cotan30 (the hypotenuse of the large triangle) is $\sqrt{3}$ and then using 3 medium triangles that tan30 is $\frac{\sqrt{3}}{3}$ and using 2 small triangles that cos3 is $\frac{\sqrt{3}}{2}$. They could also see that cos is on the horizontal axis and sin on the vertical axis and also the tan axis is vertical, and the cotan is horizontal. They still need a lot more work to become comfortable with these concepts, but I believe they have a good start now. We will continue the work probably for 2 more sessions as this is a good topic to master for them now.

sinus (sin), cosinus (cos), tangent (tan) and cotangent (cotan) of the 30° Angle and 60° Angle in a 30-60 Right Triangle			
30° Angle		60° Angle	
Ratio of Opposite side to Hypotenuse sin(30)		Ratio of Adjacent side to Hypotenuse Cos(60)	
S	$\frac{1}{2}$	S	$\frac{1}{2}$
M	$\frac{1}{2}$	M	$\frac{1}{2}$
L	$\frac{1}{2}$	L	$\frac{1}{2}$
Ratio of Adjacent side to Hypotenuse Cos(30)		Ratio of Opposite side to Hypotenuse sin(60)	
S	$\frac{\sqrt{3}}{2}$	S	$\frac{\sqrt{3}}{2}$
M	$\frac{\sqrt{3}}{2}$	M	$\frac{\sqrt{3}}{2}$
L	$\frac{\sqrt{3}}{2}$	L	$\frac{\sqrt{3}}{2}$
Ratio of Opposite side to Adjacent Side tan(30)		Ratio of Adjacent side to Opposite Side cotan(60)	
S	$\frac{\sqrt{3}}{1}$	S	$\frac{\sqrt{3}}{1}$
M	$\frac{\sqrt{3}}{1}$	M	$\frac{\sqrt{3}}{1}$
L	$\frac{\sqrt{3}}{1}$	L	$\frac{\sqrt{3}}{1}$
Ratio of Adjacent side to Opposite Side cotan(30)		Ratio of Opposite side to Adjacent Side tan(60)	
S	$\sqrt{3}$	S	$\sqrt{3}$
M	$\sqrt{3}$	M	$\sqrt{3}$
L	$\sqrt{3}$	L	$\sqrt{3}$



Math Circle 10 01/10/2023 2:00 to 3:00 pm

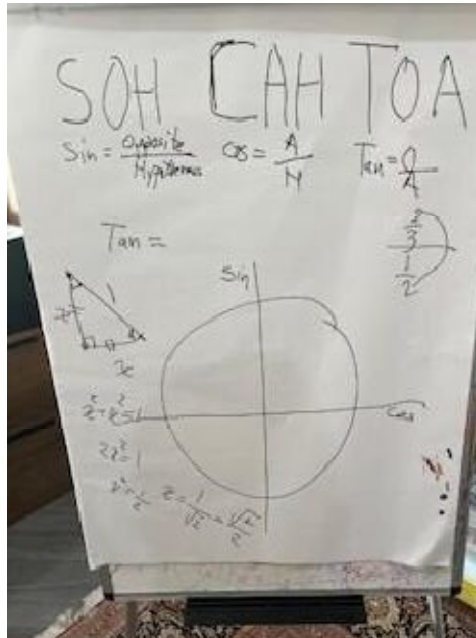
Cyrus, Marc, and Dar (Big Cyrus my son observed)

Before we started, I told them about SOH CAH TOA that is used in schools as shorthand for students to remember the definition of Sin, Cos and Tan.

Sin = Opposite/Hypotenuse (SOH)

Cos = Adjacent/Hypotenuse (CAH)

Tan = Opposite/ Hypotenuse (TOA)



Then we continued with the activities from the book Exploring Trigonometry with Dr. Super's Trigrams. I had modified the activities so they would draw sin and cos graphs using the values from the Trigonometric Circle working together.

They ran into trouble as we had not covered the sin and cos for the 45-degree angle. I showed them how using the Pythagorean theorem they can calculate the $\sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$. Then they completed the Table that had all the angles from 0 to 360 both in degrees and radians.

Marc was writing much slower and carefully and Cyrus was writing a little faster. Yay for both. The improvement in the handwriting for Marc is remarkable and he must be praised for it.

We also discussed the relation between degrees and radians and how 360 degrees was equivalent to 2π radians. But this will need some more discussion. They used what they had done in the last session for angles less than 90 degrees and continued with some help from me to find the sin for all the important angles

from 0 to 360 degrees. 0, 30, 45, 60, 90, 120, 135, 150, 180, 210, 225, 240, 270, 300, 315, 330, and 360.

Cyrus

Appendix 5a. Values Template for Students

θ Degrees	θ Radians	Sin θ	Cos θ	Tan θ	Cot θ
0	0	0	1	0	∞
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	∞	0
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
180	π	0	-1	0	∞
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
270	$\frac{3\pi}{2}$	-1	0	∞	0
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
360	2π	0	1	0	∞

45 Exploring Trigonometry with Dr. Super's TI-84/84 Plus

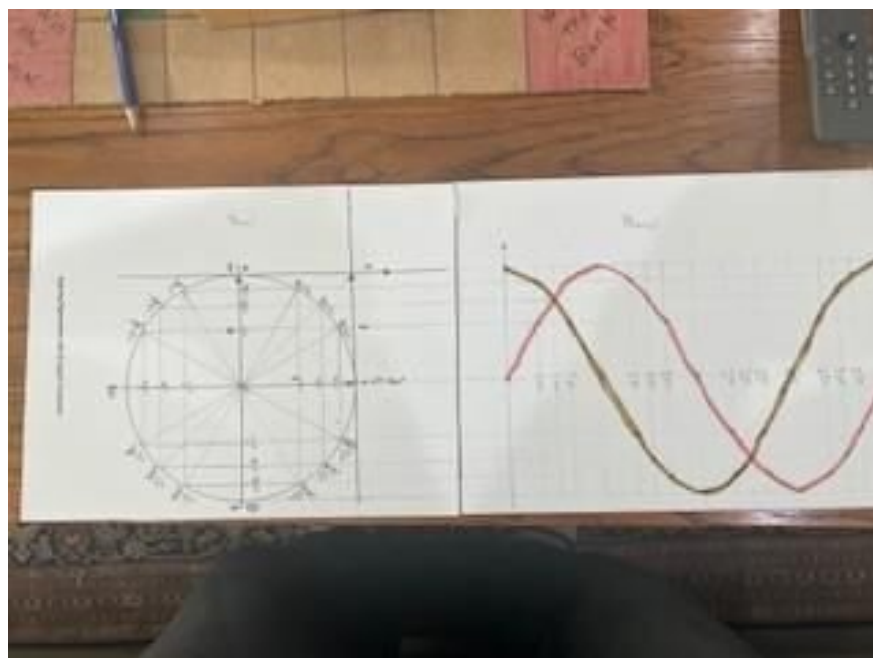
Marc

Appendix 5a. Values Template for Students

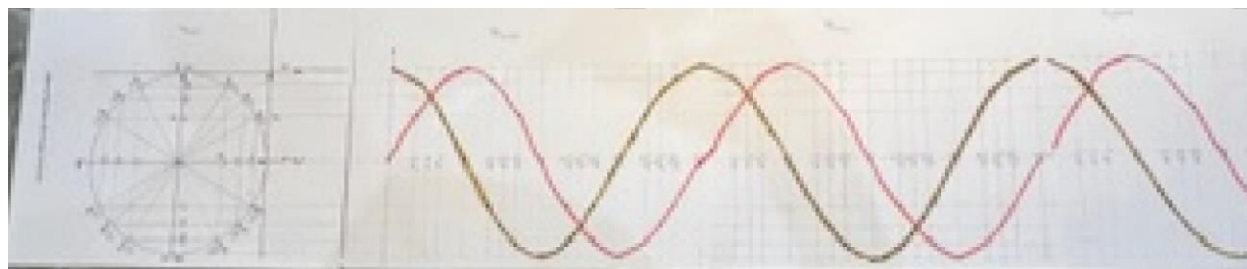
θ Degrees	θ Radians	Sin θ	Cos θ	Tan θ	Cot θ
0	0	0	1	0	∞
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	$\frac{\pi}{2}$	1	0	∞	0
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
180	π	0	-1	0	∞
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
270	$\frac{3\pi}{2}$	-1	0	∞	0
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
360	2π	0	1	0	∞

45 Exploring Trigonometry with Dr. Super's TI-84/84 Plus

Then Marc read off the values of the different angles and Cyrus put them on the chart that I had prepared for them and then connected the points and also went over it with a red pen. Next, they completed the table for the cos and this time mark placed the points on the graph and colored it gold. They clearly understood how the values were changing and both got ahead of the person who was reading off the values when they put the points on the chart.



Next, I asked them what they thought these curves will do as we continued around the circle and they both said that the same pattern will continue. I made copies of the page, and we taped them to see that the patterns for sin and cos indeed repeated. It was clear that they were impressed by this, and both had eyes that had lid up.



Next Cyrus wanted to look at the Graphing Calculator Desmos on the web that I had shown him before for the equation of the line, and Marc said that Hugo is also using it. So, we look at some of the examples and talked about amplitude and frequency and I told them that almost all the oscillating waves like light and electro-metagenetic waves are made by combining sin and cos waves and there is a branch of math called Fourier Series that does this. I will try to find a relatively simple You Tube video for them to watch next time after they complete the tan and cotan curves.

I think this was one of our best sessions and as the material is covered in advanced math courses in high school or college it was very impressive how they got it and got excited about it. I put in a lot of time so they could do the activity in 1 hour and it came off really well;=>

Math Circle 11-12 Trigonometry 01/24/2024 & 01/31/2024 1:00 to 2:00 pm

Cyrus, Marc, and Dar

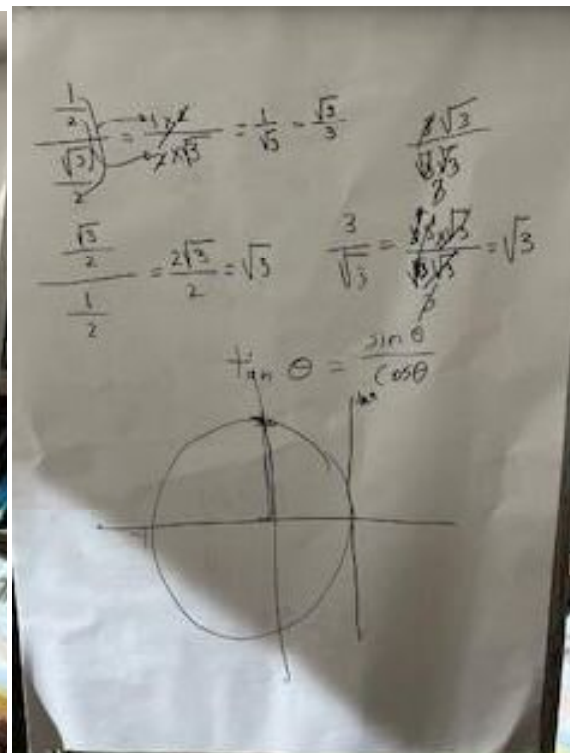
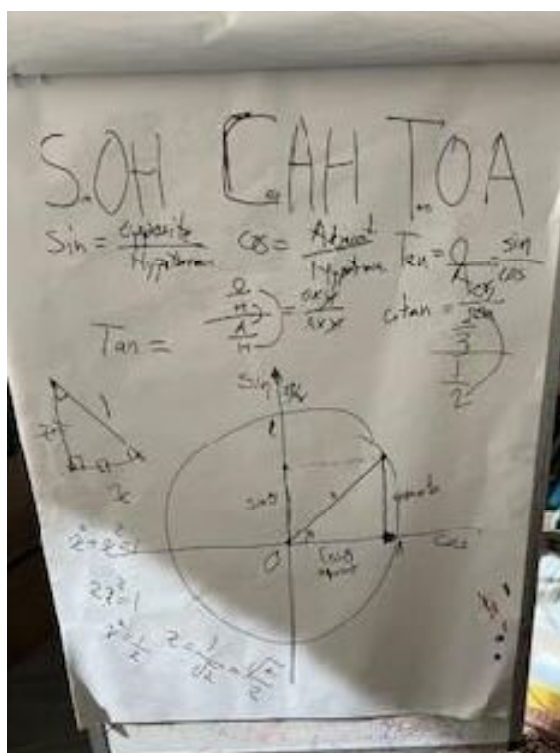
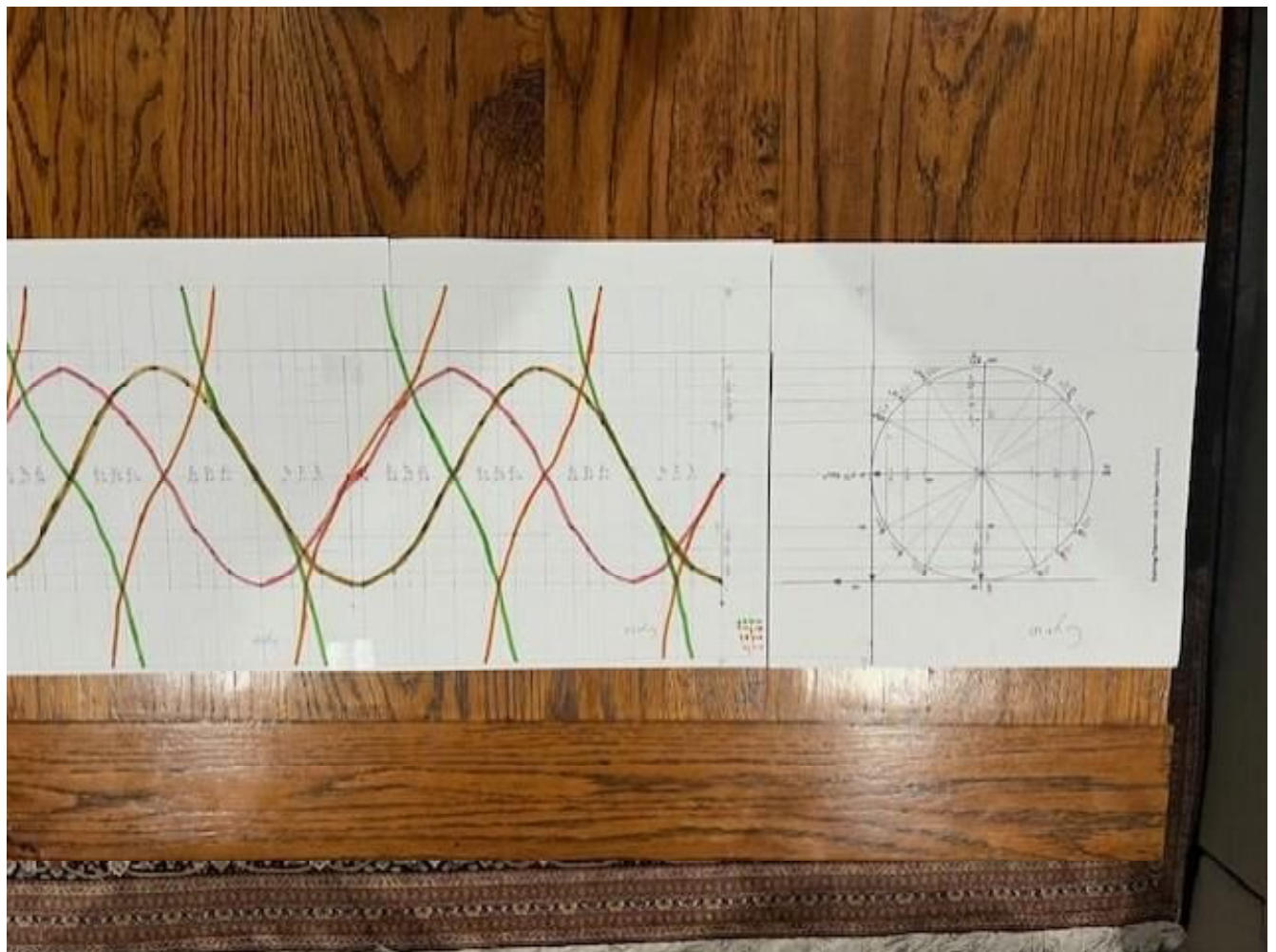
We continued with the activities from the book Exploring Trigonometry with Dr. Super's Trigrams. I had added some activities for these sessions initially they completed the table for all the angles from 0 to 360 degrees in 30-degree intervals using the Trig circle that I had marked for them.

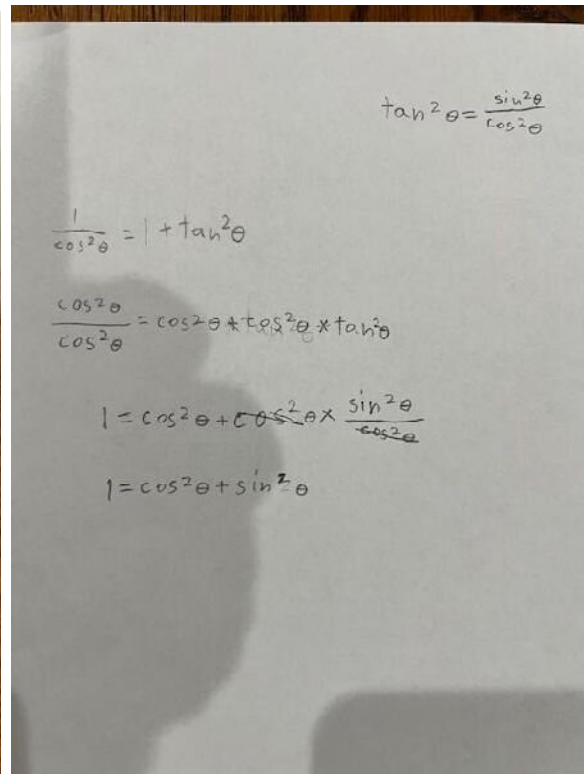
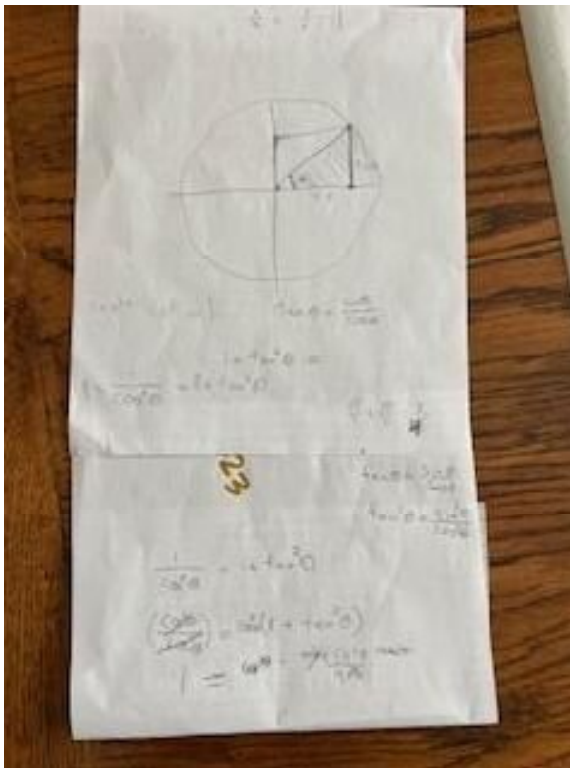
In the first session we reviewed the SOC CAH TOA and also talked more about the sin and cos for the 45-degree angle. They kept on working on all the angles to find the tangent and cotangent. When they were done with each one, they would draw it in pencil on the big graph and then color it and make a color legend. As usual Marc went fast and finished first, but he was not as careful as last time but had much better penmanship than way before. Cyrus also took his time and completed the cotan drawing the next afternoon. We had a good discussion on the meaning of the tan and cotan going to infinity at 0, 90, 180, 270 and 360 and they could see that they would go to a very positive number first and then would shift quickly to a negative large number. They both used the circle well and saw the patterns that were repeating for all the angles. Once Marc was done, I gave him the 2 identities $\sin^2\theta + \cos^2\theta = 1$ and $1/\cos^2\theta = 1 + \tan^2\theta$. I did the same with Cyrus the next afternoon. Cyrus complained semi jokingly and said why are you doing the trig identities with Marc and not with me. I said well he was done yet and we will do it when he was done. This session went a bit over an hour, but they were absorbed in it, and liked their graphs.

Cyrus

Appendix 5a. Values Template for Students

θ Degrees	θ Radians	$\sin \theta$	$\cos \theta$	$\frac{\sin \theta}{\cos \theta}$ Tan θ	$\frac{\cos \theta}{\sin \theta}$ Cot θ
0	0	0	1	0	∞
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
90	$\frac{\pi}{2}$	1	0	∞	0
120	$\frac{\pi}{2} + \frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$
135	$\frac{\pi}{2} + \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1
150	$\frac{\pi}{2} + \frac{\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$
180	π	0	-1	0	$-\infty$
210	$\pi + \frac{\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
225	$\pi + \frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1
240	$\pi + \frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
270	$\frac{3\pi}{2}$	-1	0	∞	0
300	$\frac{3\pi}{2} + \frac{\pi}{6} \left(\frac{5\pi}{3} \right)$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$
315	$\frac{3\pi}{2} + \frac{\pi}{4} \left(\frac{7\pi}{4} \right)$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1
330	$\frac{3\pi}{2} + \frac{\pi}{3} \left(\frac{11\pi}{6} \right)$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$
360	2π	0	1	0	∞



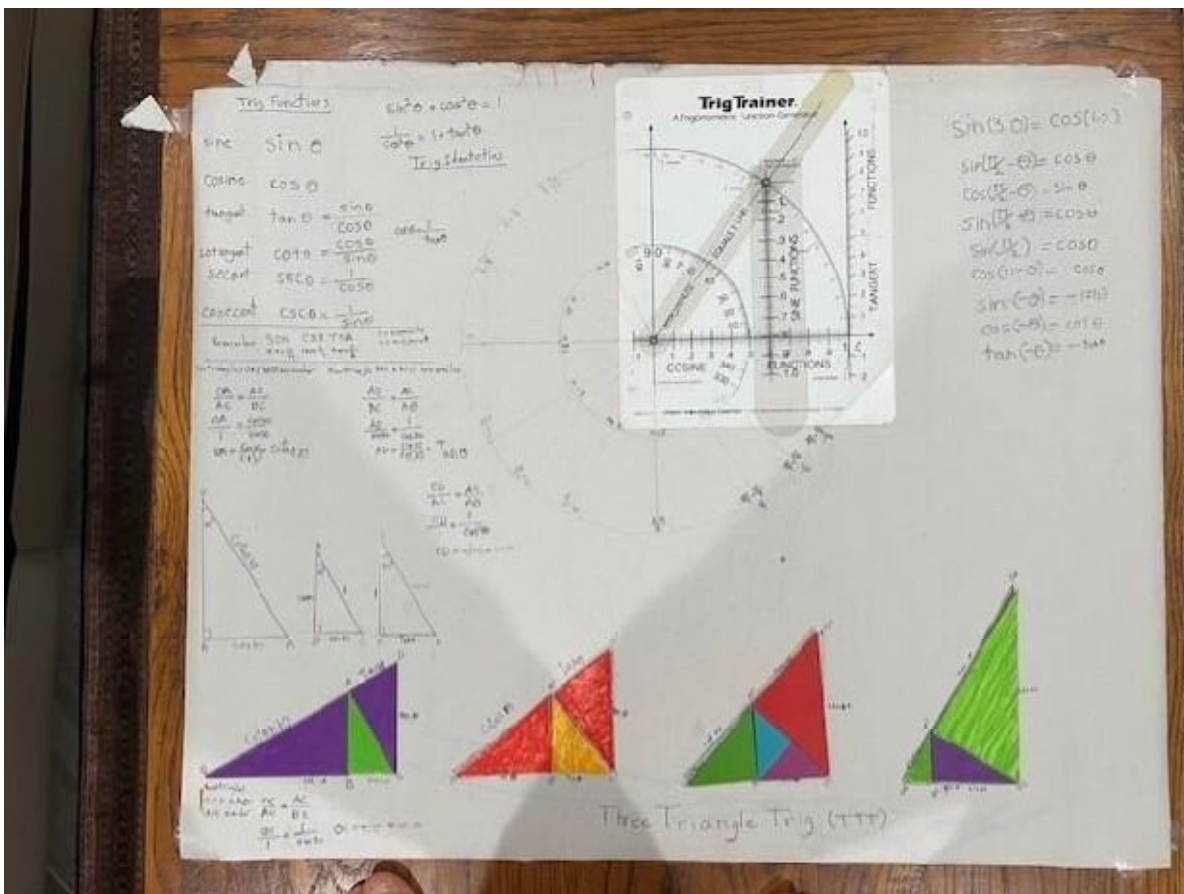
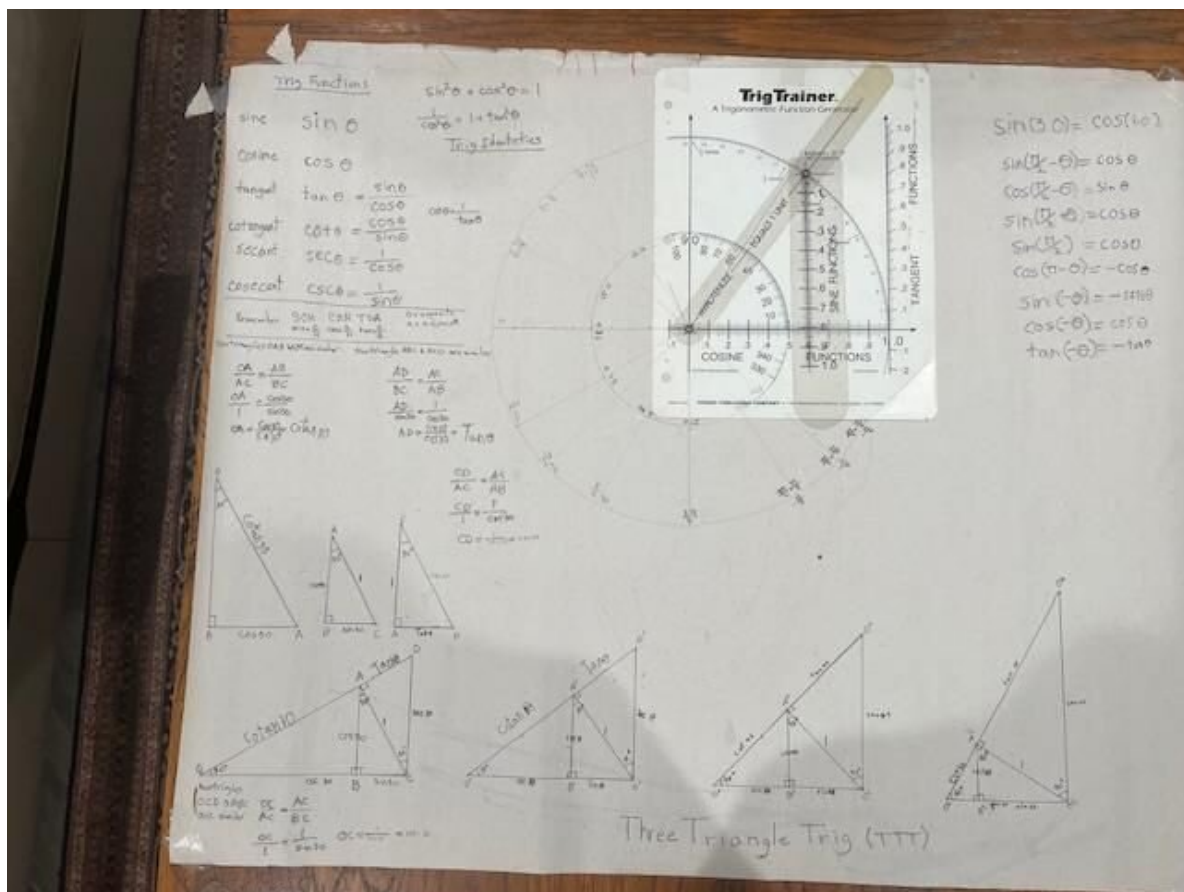


Marc and Cyrus's work on Trig identities with my help

In the session on the 31st I had prepared a big piece of newsprint paper where they would work on the Three Triangle Trig (TTT) method of finding all the trig functions and also on the trig circle using the circle from the big Trig Trainer that has plastic arms that show the values for sin and cos of any angle up to 90. I made the circle bigger so they could read all the angles for 0 to 360.

First, we went over the trig functions and the fact that cotan was really the inverse of tan and this meant that $\cotan = 1/\tan$ (this was not evident for them even though they knew what the definition of each one was). Next, we talked about the definition of secant and cosecant and the fact that they were inverses of cos and sin.

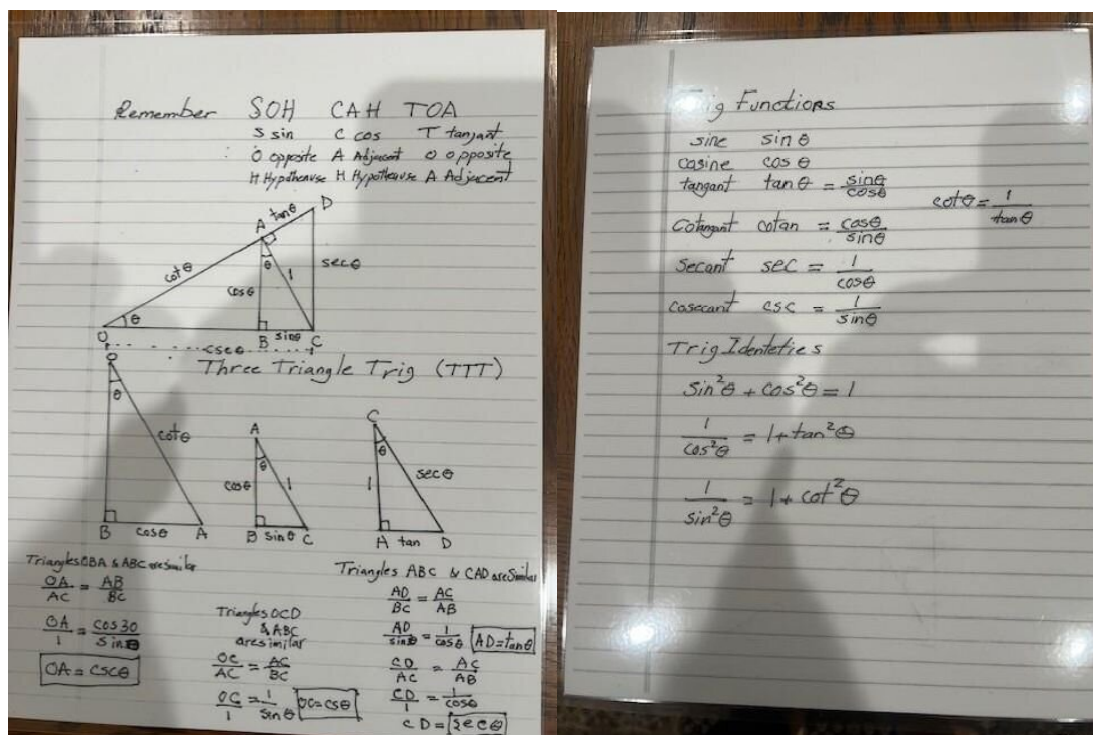
Cyrus arranged the 3 triangles for the different angles (as he had seen it yesterday and done it) and Marc helped him. These hypotenuse of the middle triangle is always 1 unit). Next, they took turns and for the 30-degree angle used similarity to find the side that was showing the tan, cotan, sec and cosec. I helped them a little but they did it mostly themselves.



Next, they marked all the trig functions on both sets of the 3 triangles related to the 30-degree angle. They took turns and did it with the other 3 triangles related to θ , 45 and 60-degree angles. We discussed the fact that θ can be any angle and as long as the hypotenuse of the middle triangle is 1 the same sides will show the trig functions. They are a bit hazy on this but they accepted it.

We then moved on to the Trig Circle and the Trig identities for $\pi/2 - \theta$, $\pi/2 + \theta$, $\pi - \theta$, and $-\theta$, each time with my help they used the Trig Circle and the plastic arms to find the right side of the identity as cos or sin of θ . Before starting we discussed the fact that the perimeter of a circle with radius r is $2\pi r$ (Cyrus said that but he was not sure what exactly the radius was, so we discussed it). Then for the trig circle this will be 2π as $r=1$. Next, I told them to use 30 degrees for θ but they should know that it would work for any angle. Still some doubt for them about θ , but this is normal at this stage. Next, I asked them which method would be better, the TTT or the Trig Circle and they both said the Trig Circle. I asked them why that would be better but before they could answer I asked them again (I should not have) can we do negative angles with TTT and they said no.

The Trig sessions went really well but it would be great if you asked them to explain what they have learned from their papers and graphs without me interfering to see how much they have retained. This was a bold undertaking for two 9-year-old kids even though they are so bright, but it went as well as I had hoped, and they really enjoyed it as I told them that most kids would study this in high school or even in college.





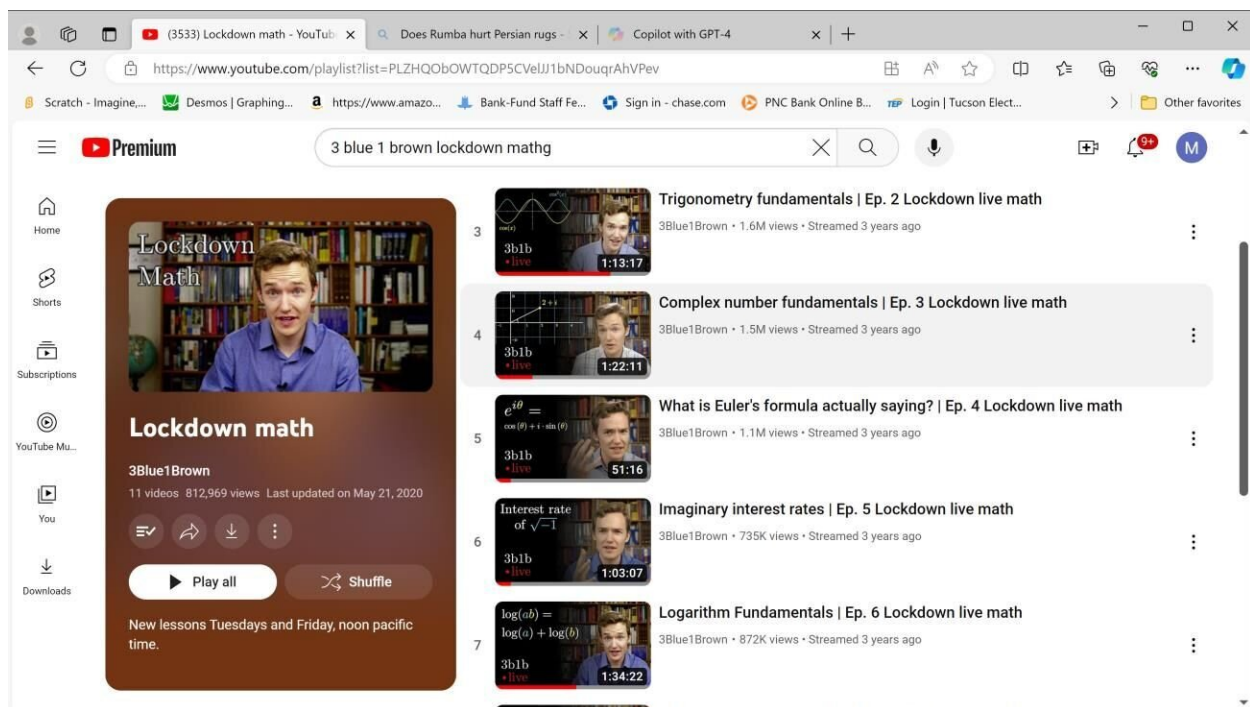
Math Circle 13 Trigonometry Lockdown Math 2/7/2024 1:00 to 2:15 pm Cyrus, Marc, and Dar

We watched the Trig video from the 3 Blue 1 Brown Lockdown Math (Live) that I have watched fully, and it is amazing. Grant Sanderson is a star developer from Khan Academy and has made several hundred videos in his own channel 3 Blue 1 Brown related mostly to math and some Physics. He is a handsome soft spoken and non-assuming personality, so it is a pleasure listening to him. All of these videos have great visualizations and graphics, and this particular series is done as a “live” with several thousand watching during COVID times and the students answer questions as the video proceeds. Of course, now it is only on You Tube. This was the third video from the series (the first is just an intro). Some of the videos get tricky but we may do the Complex Numbers, Euler’s formula, and the ones on Logarithm fundamentals and Natural Logarithm if Cyrus and Marc keep their focus as they did on this one.

Cyrus asked a lot of questions and I stoped the video each time and tried to answer them. Marc was very focused but more quiet but got more of the questins right in particular at the beging when they had to guess at some of the answers. As we got to the parts that we had covered the weeks before they both did really well and answered all the questions correctly. I really think that this is the way of the future for the kids to learn all we need is a half dozen Grant Sanderson’s in different scientefic fields for the whole student population in the World!.

The session went for an hour and ten minutes. After a half hour I suggested that we take a break but they refused and said they wanted to continue. Cyrus said he was hungry so I got them vegtebles and food and we continued another 40 minutes with several stops and discussion. Most of the topics in this video we had covered in the Math Circles the previous weeks but I think now they have a much better handle on Trigonometry.

These same videos will be great for Hugo in particular the ones that I mentioned above.



<https://www.youtube.com/live/yBw67Fb31Cs?si=uHD47STPiyqmjT15>

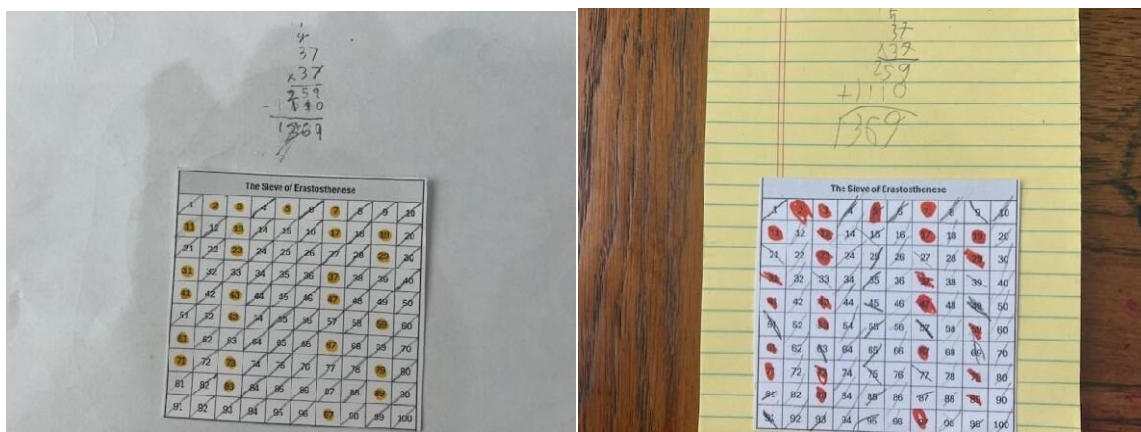


They are so good together. Someday not in too distant future when they are great Mathematicians they will look at these notes and enjoy them and what we did together.

Math Circle 14 Complex Numbers 2/14/2024 1:00-2:15

Cyrus, Marc and Dar

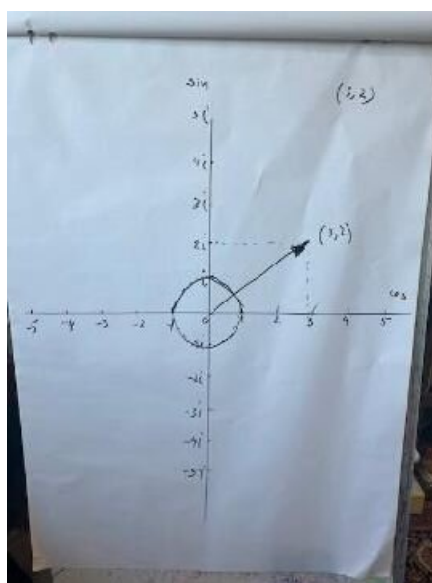
We started with a challenge activity “The Sieve of Eratosthenes” in which they crossed out multiples of primes starting with 2 to find all the prime numbers on a table of numbers up to 100.



As usual Cyrus took his time and Marc was off to the races even before I finished completing the instructions. However, they both completed the task and found all the 25 primes after a few counting and checking. Next, I asked them what prime number they had to go up to and cross out it's multiples to find the rest of the primes and they both said 7. I asked why they did not have to go to 11 and after some discussion they end up with the fact that 11×11 was 121 and so it was not in the table (and all the other multiples of 11 were already crossed out – I am pretty sure they made this conclusion even though they did not say it).

Next came the challenge: What prime do you have to go up to and cross out it's multiples to find all the primes up to 1000. I left the room and encouraged them to discuss it. I came back after 7 minutes, and they were not done and had stopped talking about it. So, I engaged them in the discussion that why they could stop at 7 to find all the primes up to 100 and tried to make them see that they could make the same reasoning for 1000. Eventually they did come up with the fact that they had to go up to 31 only as $37 \times 37 = 1369$ and 31 was the prime preceding 37.

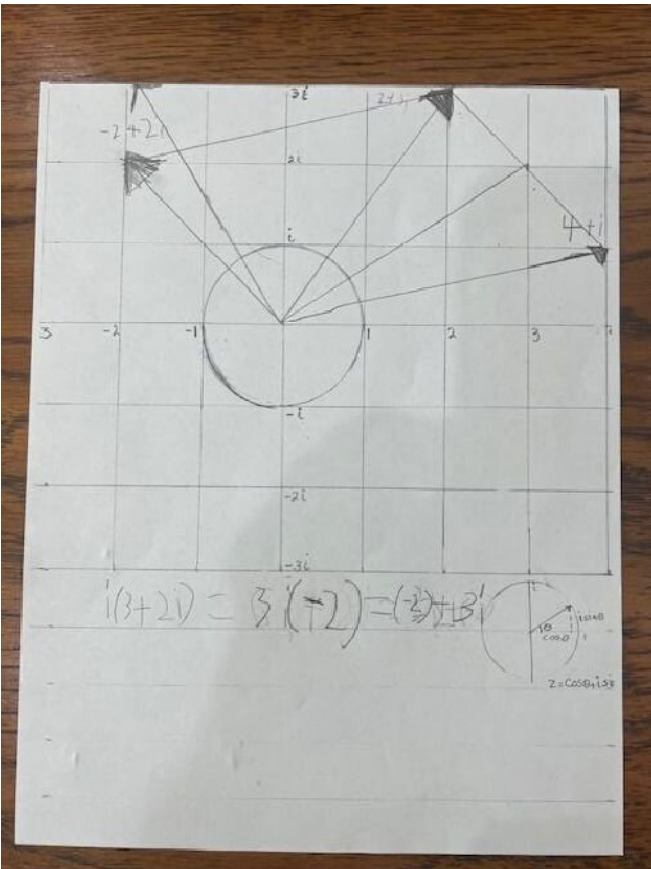
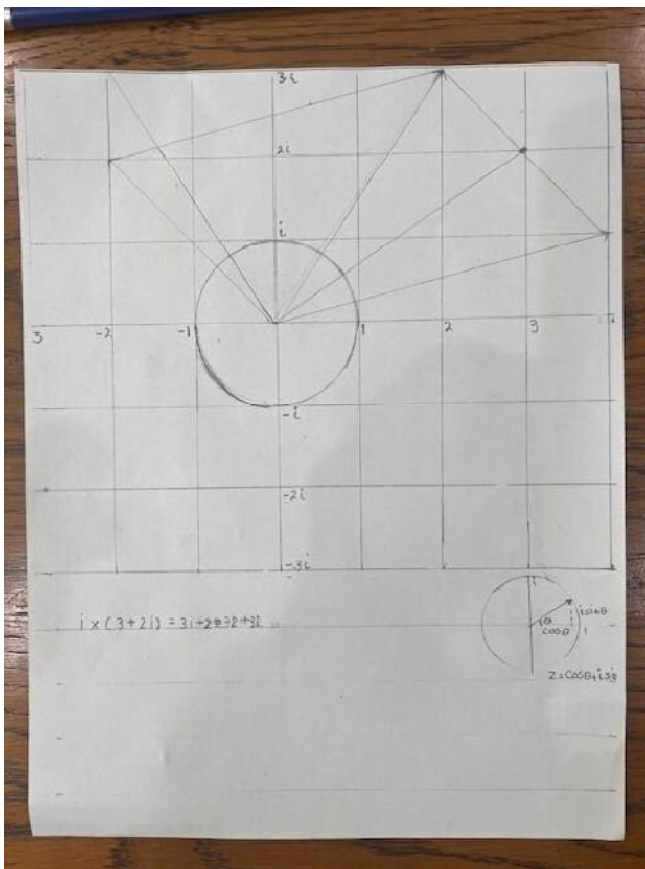
Next, I told them about vectors and how if we connected the origin to any point on a the coordinate system we would get a vector. Also that we were going to watch the [Complex Number Video](#) and this will connect complex numbers with rotation and trigonometry when the y-axis becomes the imaginary axis! They did look puzzled at this point.



We covered 22 minutes in the video with a few stops and talked about what was discussed. They answered a couple of the questions and understood that addition of Complex numbers is simple and they only had to add the real parts and the imaginary parts. They both agreed that 2, square root of 2, -1, infinity and i were all real numbers in the sense that they existed with most of the students in the live.

For the multiplication of complex vectors, they understood that multiplying by i turns the vector 90 degrees counter clockwise. We were looking at multiplying by an arbitrary complex vector $z=a+bi$ that we had to stop as we had done an hour and 15 minutes of a very difficult subject.

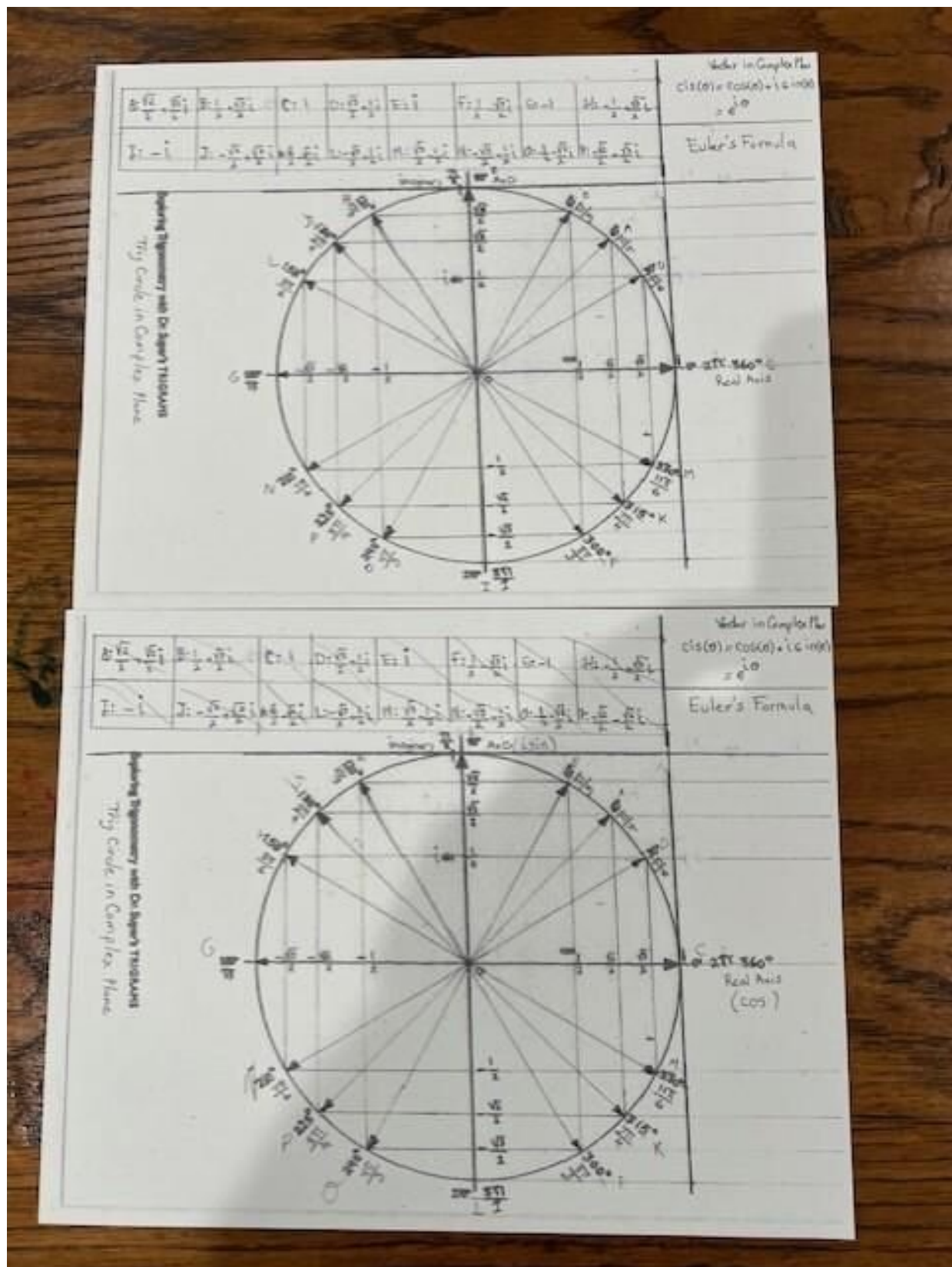
I am very impressed with how they followed everything. We will complete this video in the next session. They went and played some Ping Pong, and I told them how they should compete but doing it nicely! That means they should try to beat the best that the other person can offer and never should just want to win at any cost. They listened but I am not sure if it sank in.



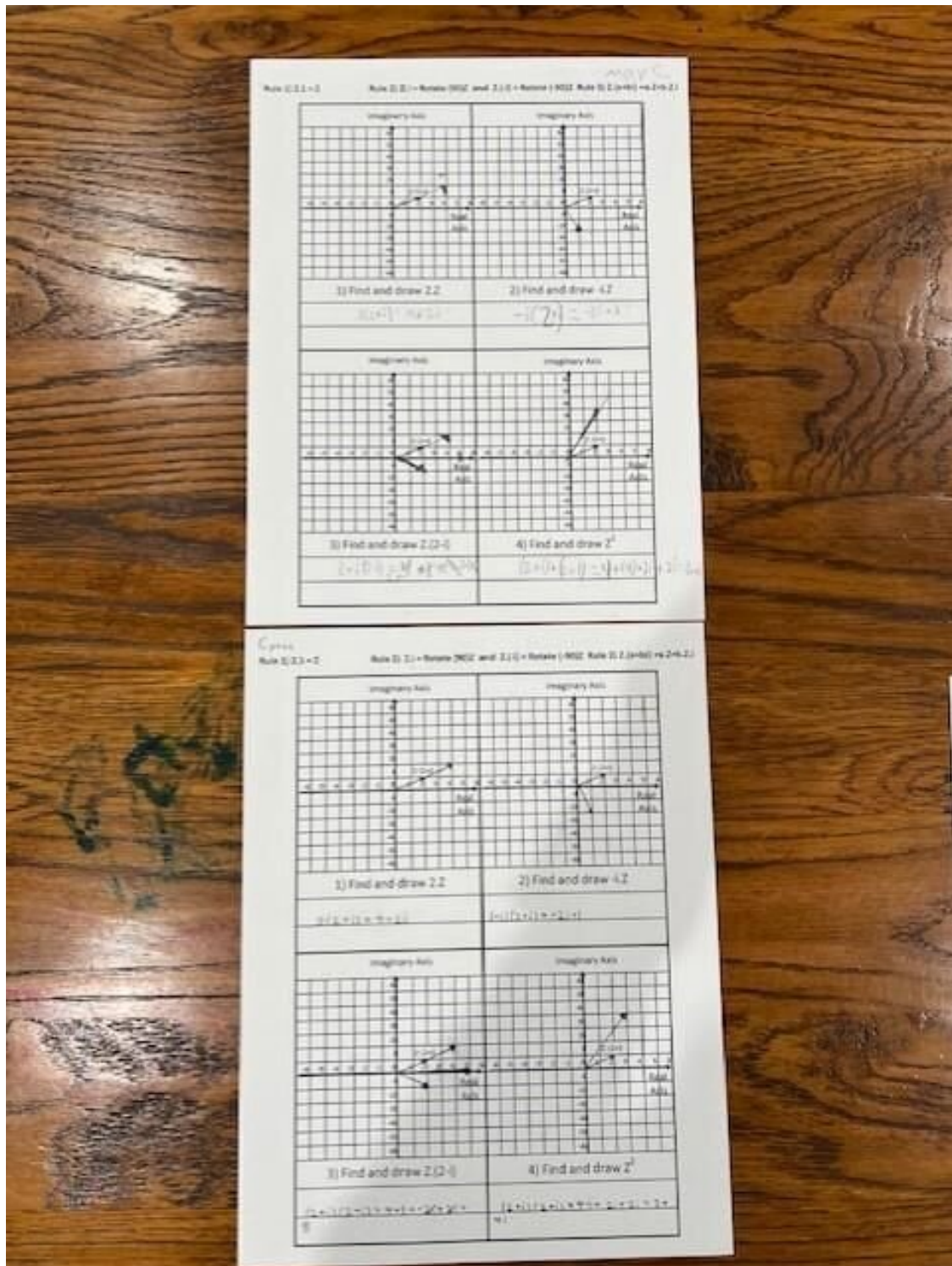
Math Circle 15 Complex Numbers 3/06/2024 1:50-2:20

Cyrus, Marc and Dar

We continued with the activities from the [3Blue1Brown on Complex numbers](#). I initially I had them do a review of the values of sin and cos on the trig circle in the complex plane as we had missed two sessions.



They marked the points on the trig circle in the complex plane corresponding to the values given by their sin and cos. Cyrus had seen a little of this when I was preparing it but they both worked together and completed it quickly, and Cyrus even complained that it was too easy! Next, I had them go through and do the exercises that are done in the video before we went back to the video so they could go through the video faster and get a better understanding of how multiplication works on the complex plane. I did break up the steps for doing the problem $(2+i)(2-i)=5$ both algebraically and by using the rules for drawing and extending and rotating the vector $2+i$ in the complex plane. There was a confusion about how $-i$ would turn the vector $2Z$ that Cyrus pointed out and I had also noticed. I will have to explain to them in the next session as it has to do with how adding vectors works in the complex plane (or anywhere). They also did the problem for Z^2 . Marc is still going fast in particular when he is not sure what the correct method is, Cyrus just stops! They need more practice with these concepts, but it is great that they get it even at this level.

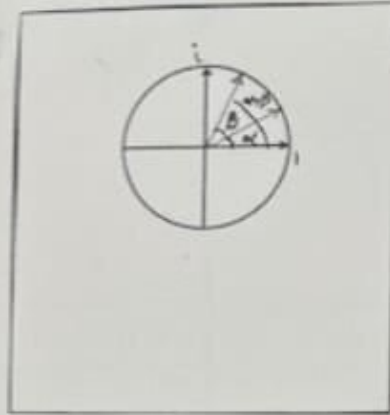


Next we watched the video and saw a bit of Payton programing to show how the rotation would really work. (We should do a little Python later). They did a good job drawing the vectors and finding the $\cos(75)$ and finding effectively the square root of $Z^2=$ and solutions for $Z^3=1$. After each problem we also watched how it was done on the video.

	
What is the complex number Z that multiplying by it has the effect of rotating by 30 degrees? Find and Draw it.	What is the $\cos(75)$? $(75=45+30)$
	$\cos(75) = \cos(45+30) = \cos(45)\cos(30) - \sin(45)\sin(30)$
	
What value of Z satisfies $Z^2=1$. Find and Draw it.	The equation $Z^2=1$ has a real solution 1. It also has two complex solutions. Find them.
$Z = \pm 1$	$Z = \pm i$
	
What is the complex number Z that multiplying by it has the effect of rotating by 30 degrees? Find and Draw it.	What is the $\cos(75)$? $(75=45+30)$
	$\cos(75) = \cos(45+30) = \cos(45)\cos(30) - \sin(45)\sin(30)$
	
What value of Z satisfies $Z^2=1$. Find and Draw it.	The equation $Z^2=1$ has a real solution 1. It also has two complex solutions. Find them.
$Z = \pm 1$	$Z = \pm i$

We ended with the formula for the cos and sin of two angles. They only examined that as I explained how it related to what they had done already and to the $\cos(2x)$ that they had seen before.

It was a great session and they lasted through it for almost a hour and a half. They needed some help with the problems, but they were able to do them with some help from Dar.



Since:

$$\cos(\alpha + \beta) + i \sin(\alpha + \beta) = (\cos(\alpha) + i \sin(\alpha)) (\cos(\beta) + i \sin(\beta))$$

check that:

$$\cos(\alpha + \beta) = \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)$$

$$\sin(\alpha + \beta) = \cos(\alpha) \sin(\beta) + \sin(\alpha) \cos(\beta)$$

Math Circle 16 Euler's Formula and Identity 3-20-2024 1:50-2:10

Cyrus, Marc and Dar

We studied Euler's Formula and saw Euler's identity. At the end we watched a short YouTube video on [Euler's Formula \(Complex Numbers\)](#).

Initially Cyrus and Marc completed the table of the cumulative sum for $1/n!$ that I had made for them (see below) and saw that as n got larger this cumulative sum got closer to Euler's Number. We had already talked about Euler's number so that had some familiarity with it.

Cyrus

Finding the Cumulative Sum for $1/n!$ (Euler's Number e)

n	n!	$1/n!$	$1/n!$ (Decimal)	Cumulative Sum
0	1	$1/1$	1.00000	1 . 0 0 0 0 0
1	1	$1/1$	1.00000	2 . 0 0 0 0 0
2	2	$1/2$	0.50000	2 . 5 0 0 0 0
3	6	$1/6$	0.16667	2 . 6 6 6 6 7
4	24	$1/24$	0.04167	2 . 7 0 8 3 4
5	120	$1/120$	0.00833	2 . 7 1 6 6 7
6	720	$1/720$	0.00139	2 . 7 1 8 0 6
7	5040	$1/5040$	0.00020	2 . 7 1 8 2 6
8	40320	$1/40320$	0.00002	2 . 7 1 8 2 8

Next they examined the table below and found the future value of compounded interest and principal for \$1 at 100% also approached the Euler's number as we compounded more and more. We spent some time discussing the meaning of the compounded interest and how it was calculated as they were examining the table.

Future Value of Compounded Interest and Principal

P Principal (\$)	k Annual Interest Rate Paid	m Number of times per period the interest is compounded	k/m Interest paid each time it is compounded	n Number of periods (typically years) or term of the loan	CumulativeSum Future Value of Compounded Interest $=P*(1+(k/m))^{(m*n)}$
1	100%	1	100.00000%	1	2.00000
1	100%	2	50.00000%	1	2.25000
1	100%	3	33.33333%	1	2.37037
1	100%	4	25.00000%	1	2.44141
1	100%	5	20.00000%	1	2.48832
1	100%	6	16.66667%	1	2.52163
1	100%	7	14.28571%	1	2.54650
1	100%	8	12.50000%	1	2.56578
1	100%	9	11.11111%	1	2.58117
1	100%	10	10.00000%	1	2.59374
1	100%	50	2.00000%	1	2.69159
1	100%	100	1.00000%	1	2.70481
1	100%	365	0.27397%	1	2.71457
1	100%	1000	0.10000%	1	2.71692
1	100%	10000	0.01000%	1	2.71815
1	100%	1000000	0.00010%	1	2.71828

Then they completed the table below and discovered that when you multiply the powers of e the powers will add up and came up with the formula.

$$e^{x+y} = e^x \times e^y$$

Property of the Powers of e

Powers of e	Decimal Value of Powers of e	Relations between Powers of e	
e^1	2.718	e^1	e^1
e^2	7.389	$e^2 = e^1 \times e^1$	
e^3	20.086	$e^3 = e^1 \times e^1 \times e^1$	$e^3 = e^2 \times e^1$
e^4	54.598	$e^4 = e^1 \times e^1 \times e^2$	$e^4 = e^3 \times e^1$
e^5	148.413	$e^5 = e^2 \times e^3$	$e^5 = e^4 \times e^1$
e^6	403.428	$e^6 = e^1 \times e^2 \times e^3$	$e^6 = e^5 \times e^1$
e^7	1096.629	$e^7 = e^3 \times e^4$	$e^7 = e^6 \times e^1$
e^8	2980.946	$e^8 = e^5 \times e^3$	$e^8 = e^7 \times e^1$
e^9	8103.047	$e^9 = e^3 \times e^3 \times e^3$	$e^9 = e^8 \times e^1$
e^{10}	22026.356	$e^{10} = e^5 \times e^5$	$e^{10} = e^9 \times e^1$
		$e^{x+y} = e^x \times e^y$	

Next we looked at the Taylor series for e^x and discussed the fact that it will look like the cumulative sum for $1/n!$ for $x=1$ that they had examined before. We also looked at the Taylor series for $\sin$ and $\cos$ and then examined the repeated powers of i to conclude end up with the Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

next they concluded for $\theta=\pi$ this would give Eulers identity:

$$e^{i\pi} = -1 \text{ or } e^{i\pi} + 1 = 0$$

They had to accept reluctantly the Taylor series expansions for e and $\cos$ and $\sin$ but they played along.

Cyrus

Taylor Series for e^x and $\sin$ and $\cos$ and Euler's Formula

$$e^x = 1 + x/1 + x^2/2 + x^3/6 + x^4/4! + x^5/5! + x^6/6! + x^7/7! + \dots$$

$$e^\theta = 1 + \theta/1 + \theta^2/2 + \theta^3/6 + \theta^4/4! + \theta^5/5! + \theta^6/6! + \theta^7/7! + \dots$$

$$\cos(\theta) = 1 - \theta^2/2 + \theta^4/4! - \theta^6/6! + \dots$$

$$\sin^{(\theta)} = \theta/1 - \theta^3/6 + \theta^5/5! - \theta^7/7! + \dots$$

$i = \sqrt{-1} \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1$

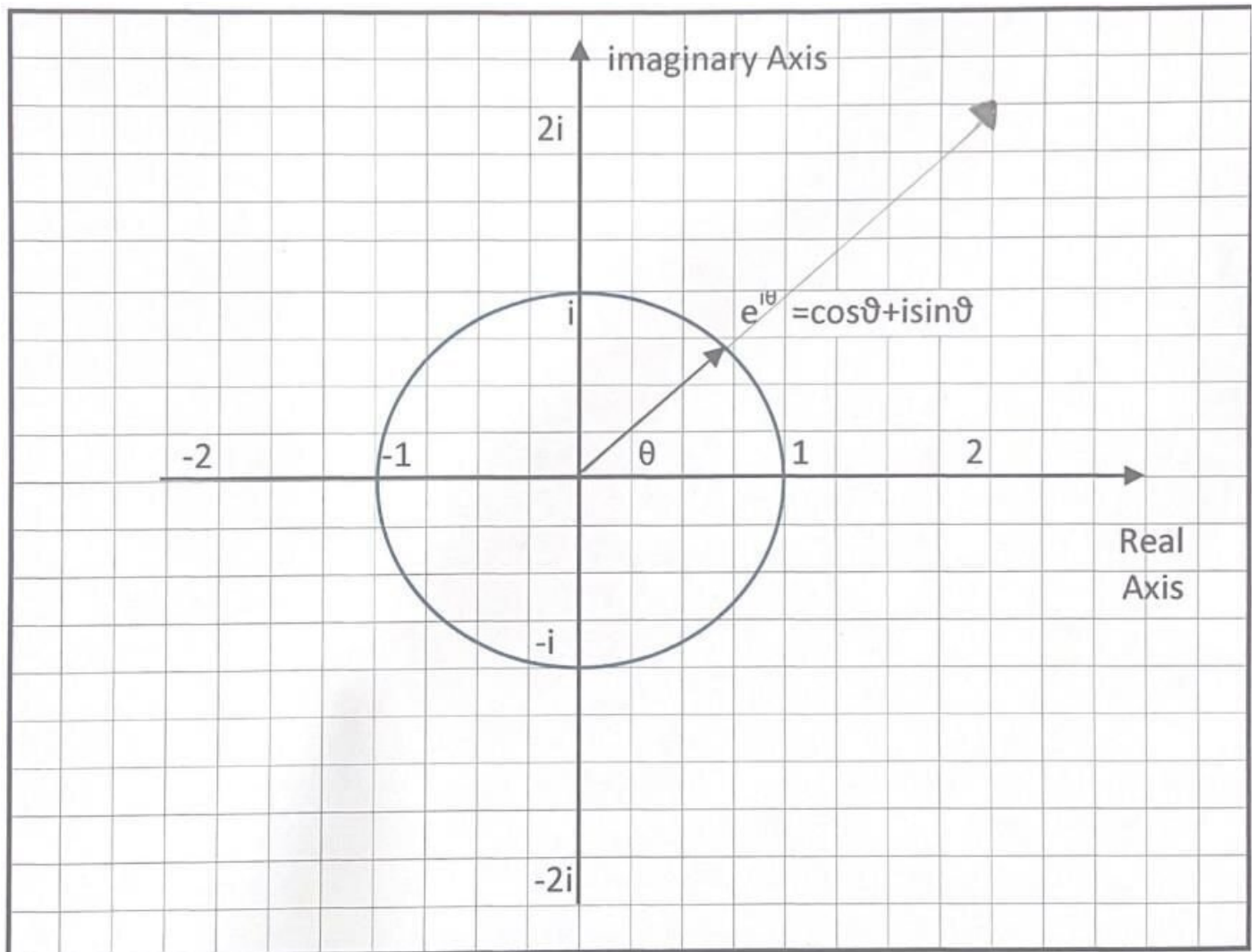
$$e^{i\theta} = 1 + i\theta/1 + (i\theta)^2/2 + (i\theta)^3/6 + (i\theta)^4/4! + (i\theta)^5/5! + (i\theta)^6/6! + (i\theta)^7/7! + \dots$$

$$e^{i\theta} = 1 + i\theta/1 - (\theta)^2/2 - i(\theta)^3/6 + (\theta)^4/4! + i(\theta)^5/5! - (\theta)^6/6! - i(\theta)^7/7! + \dots$$

$e^{i\theta} = \cos \theta + i \sin \theta$

$e^{i\pi} = \cos \pi + i \sin \pi = -1$

Finally they drew the vector $(2+2i)$ on the complex plane and we discussed that $\theta=\pi/4$.



In the end we watched the video [Euler's Formula \(Complex Numbers\)](#) That had some great animation of Euler and Bernoulli and reviewed all that they had done in the session to find Euler's Formula and Identity. This was a good review and gave them more insight to what they had done. Never the less, This session was probably way too advanced for them but they like this kind of stuff and asked me when we are going to do Calculus!

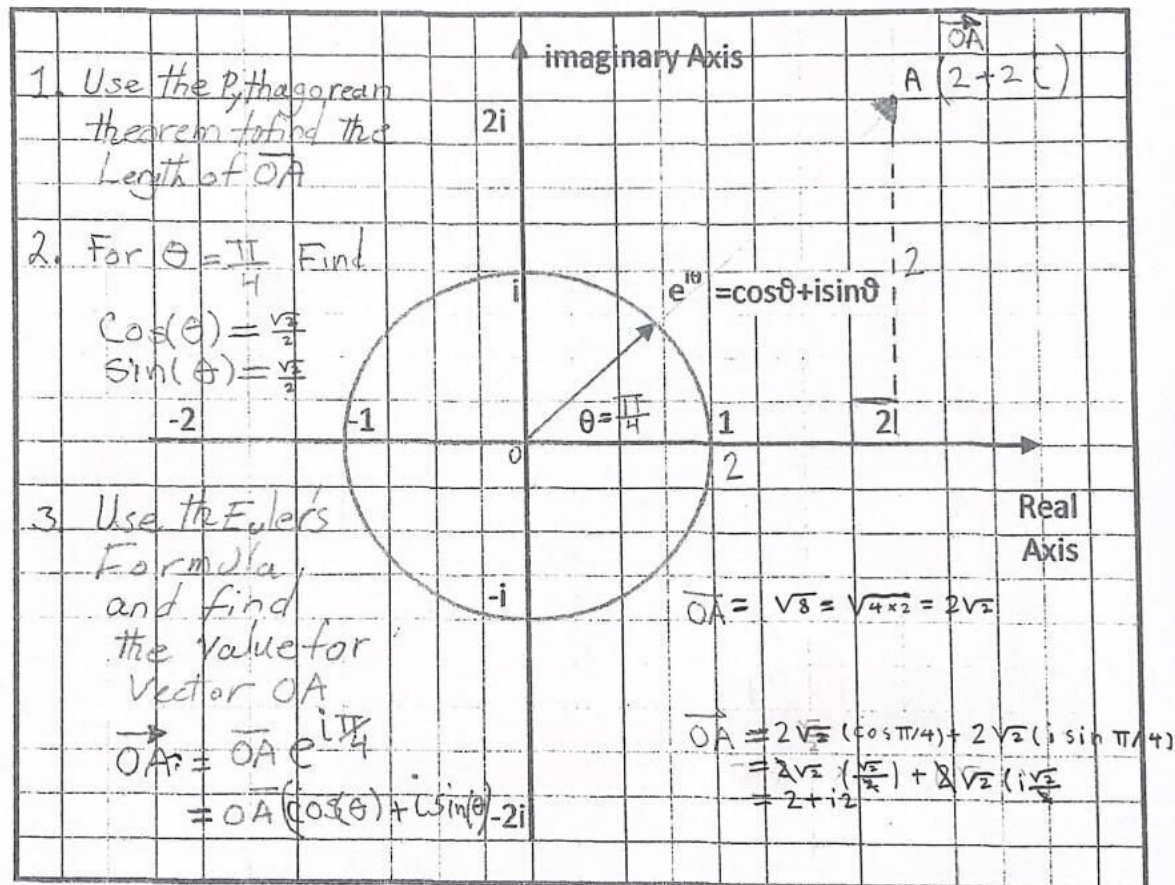
Math Circle 17 Euler's Formula and Identity 3-20-2024 1:50-2:05

Cyrus, Marc and Dar

We started by reviewing what Cyrus and Marc had drawn in the last session. They had found that the angle on the complex plane was $\theta = \pi/4$ and the point was $(2+2i)$. I helped them establish that if they used the Euler's formula they would get the same point. First they found the length OA to be $\sqrt{8} = 2\sqrt{2}$ then they wrote the

$$\text{Vector } OA = 2\sqrt{2}e^{i\pi/4} = 2\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$$

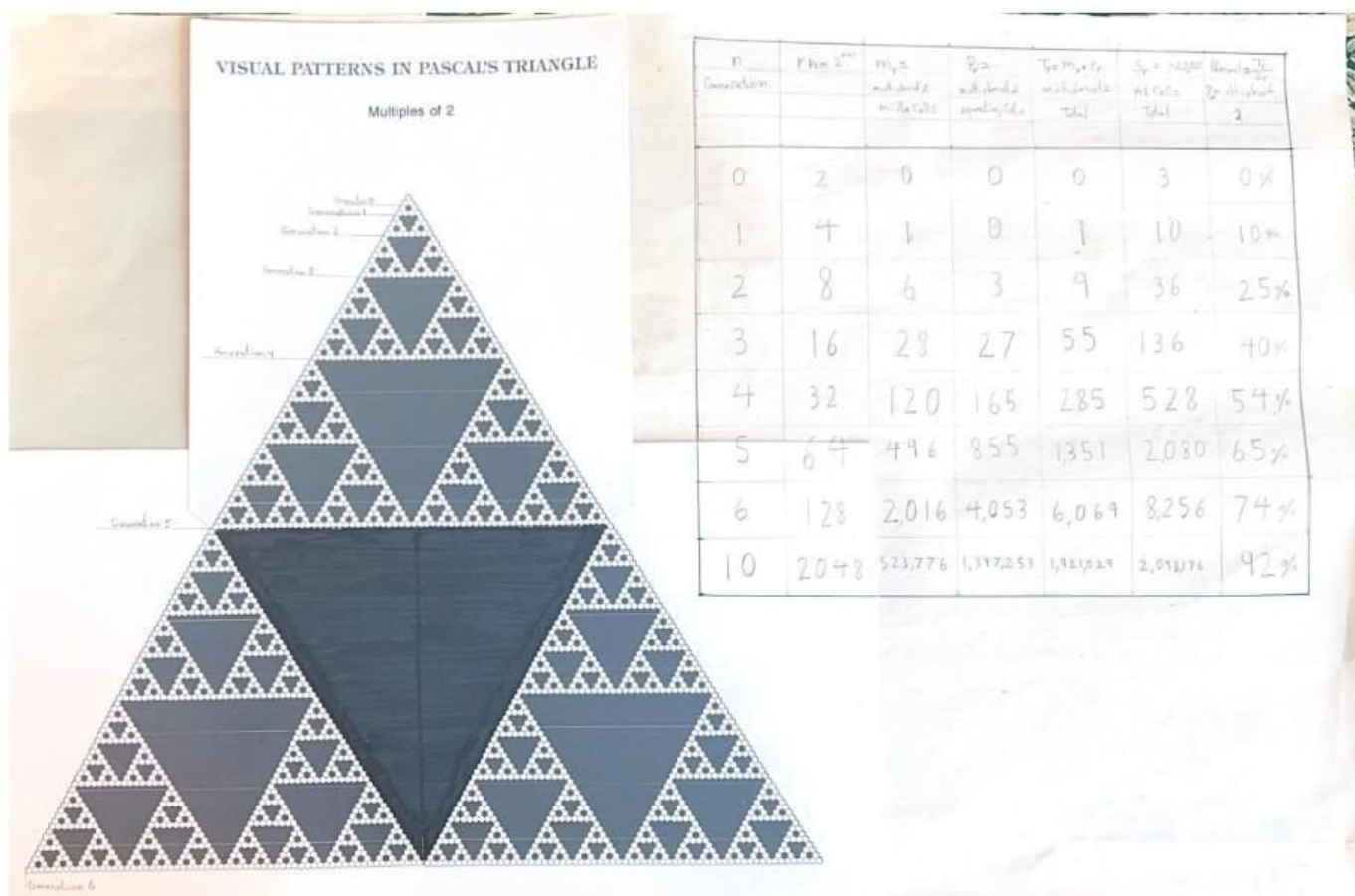
They needed a little help but they did figure out that the vector will be $(2+2i)$



$$\begin{aligned}\sqrt{8} &= \sqrt{4 \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \\ \sqrt{2}\sqrt{2} &= 2\end{aligned}$$

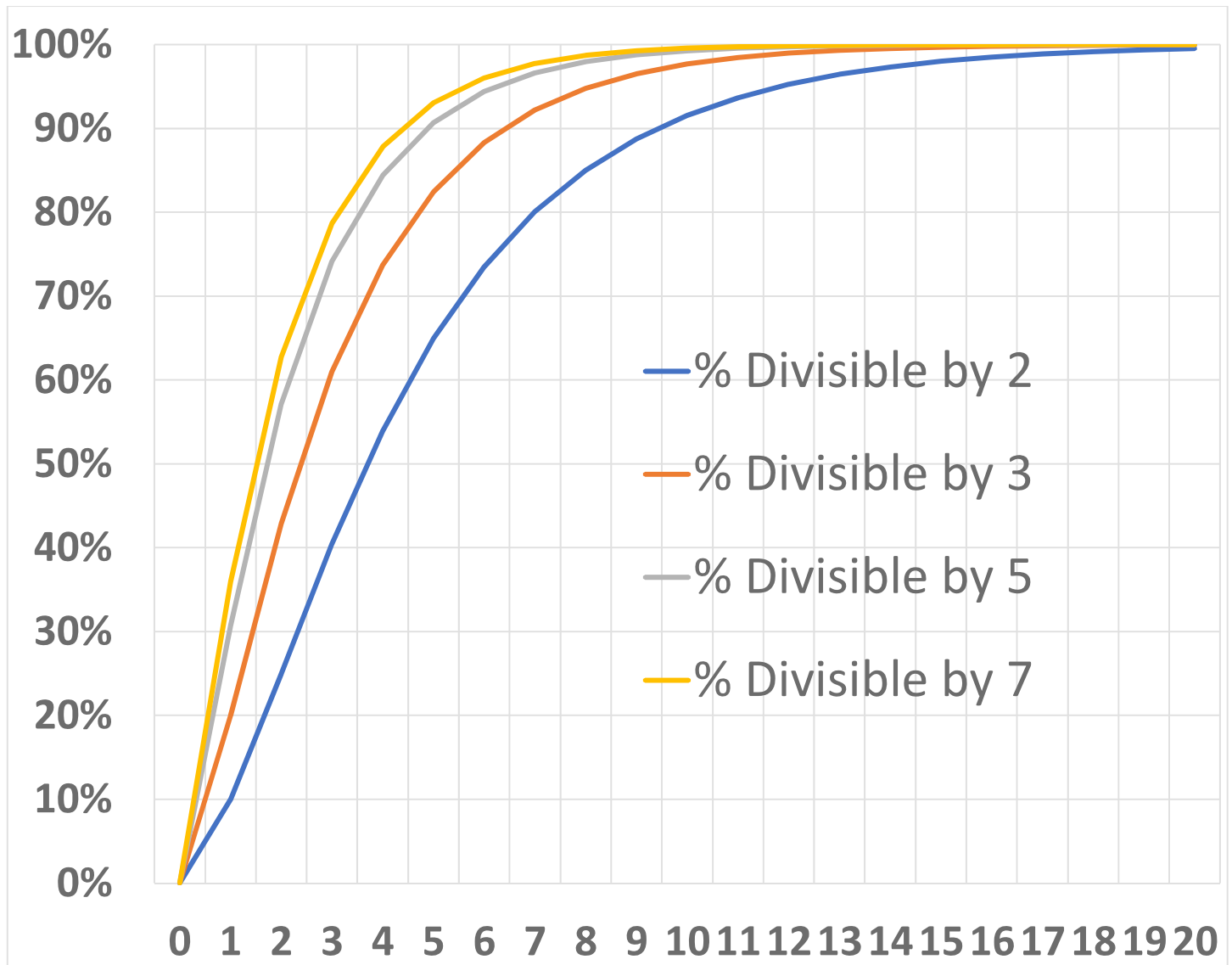
Next, we discussed the fact that in the Visual Patterns in Pascal's Triangle Dale Seymour suggested that almost all the cells will be divisible by 7 after a few generations. I told them that we were going to see if this worked for 2 and then see if we could propose that this worked for all primes or even all numbers if you went further and further in the Pascal's Triangle.

I gave them the sheet that had an image of the generations 0 to 6 for all the cells divisible by 2 (These cells were black). Marc was to do the calculations and Cyrus was to complete the table and I was the manager. This went really well and soon they realised that the formula for sum of the numbers up to n ($S = n(n+1)/2$) had to be used repeatedly and they started finding the other formula's that were necessary to complete the table. They were having a good discussion and this was great team work. By the 6th Generation 74% of the cells were divisible by 2. I also gave them the 10th Generation percentage that becomes 92%. They could also see from the beginning that the black areas will get bigger and bigger.

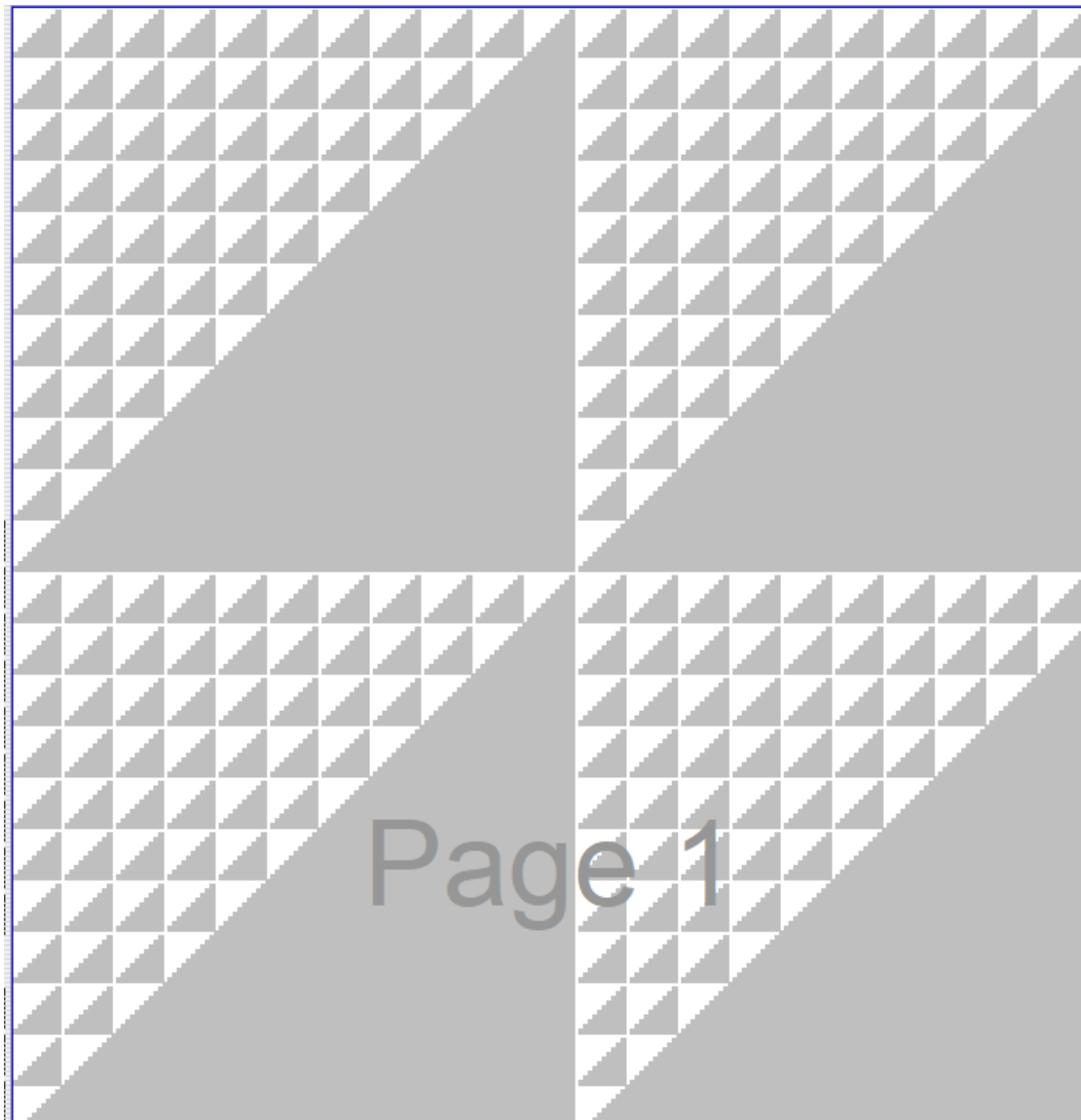


I also showed them how I did the same work in Excel for 2, 3, 5, 7, 11 and 13. I even told them that there seems to be a mistake in the book about the percentage that will be divisible by 7 at Generation 6 and the figure should be 92% and not 99.9% (Which will happen much later).

I also showed the chart showing in what generation you will get to 100% and even though it seems that 2 will get there later than the other numbers in reality for each higher number you need to look at a much larger part of the Pascal's Triangle to get to 95% or 99% divisibility by that number.



I also showed them some of the formula's that were used in Excel that were identical to what they had discovered. I even showed them the visuals for 11 and 13 and marc really wanted to see more and more of it.



Finally they watched second video of the [Calculus Series 3 Blue and 1 Brown about the derivative](#) and discussed it a little.

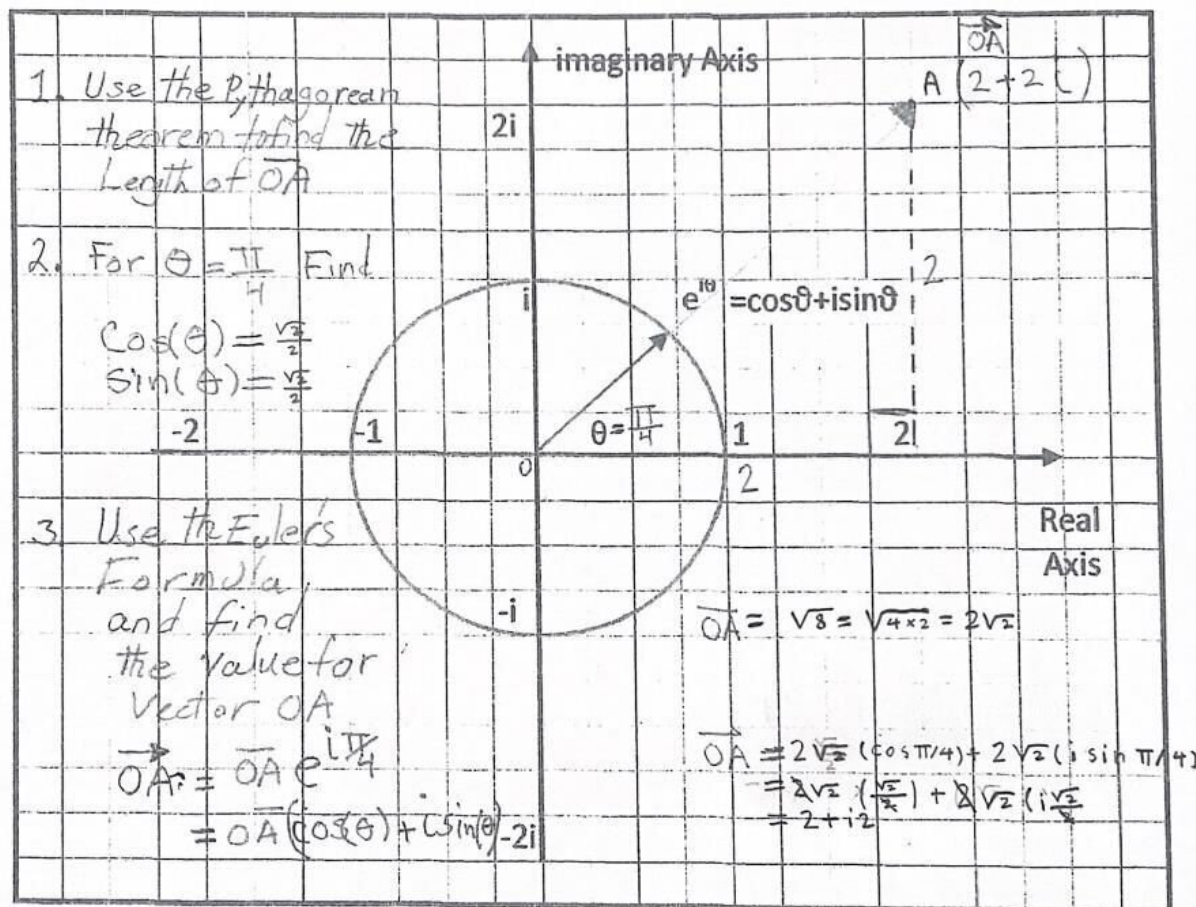
This probably the best session we have ever had and they were both excite about the discovery and we will examine it next time also. We even got some Foosball done at the end with Hugo.

Math Circle 17 Euler's Formula and Identity 3-20-2024 1:50-2:05

Cyrus, Marc and Dar

We continued again by reviewing what Cyrus and Marc had drawn in the last session below. They had found that the angle on the complex plane was $\theta = \pi/4$ for the vector OA and the point was at $(2+2i)$. So, they found that Euler's Formula would also give them the same point.

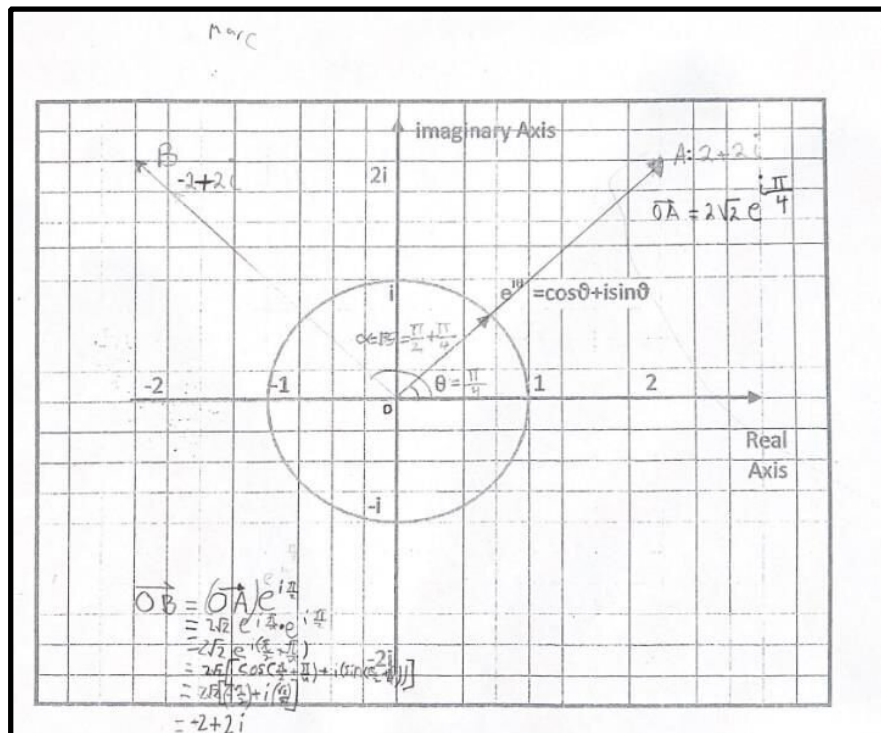
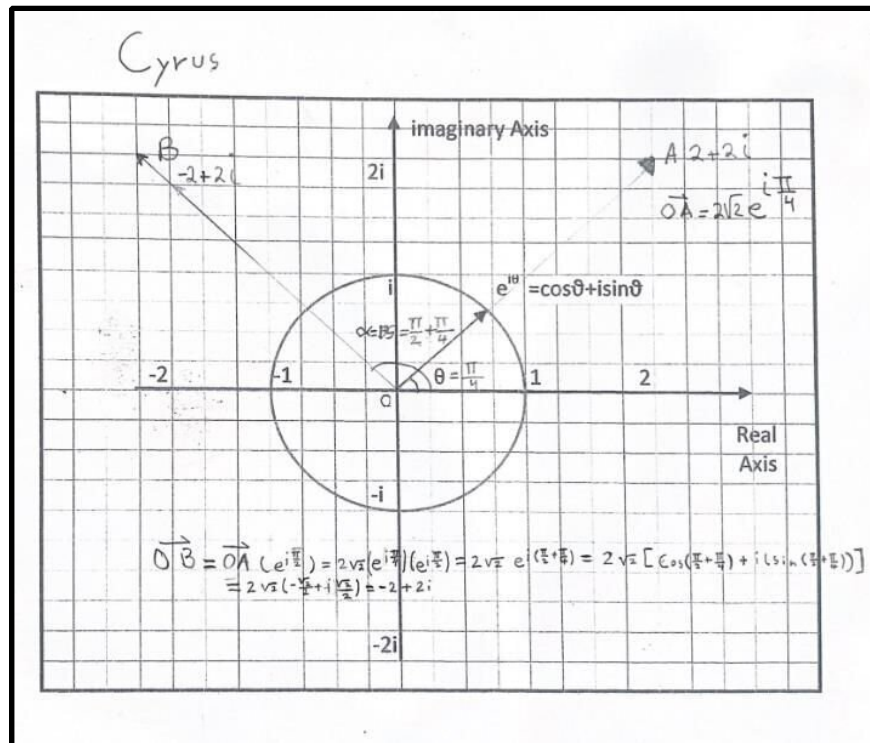
$$\text{Vector OA} = 2\sqrt{2}e^{i\pi/4} = 2\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)) = 2+2i$$



$$\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \sqrt{2} = 2\sqrt{2}$$

$$\sqrt{2} \sqrt{2} = 2$$

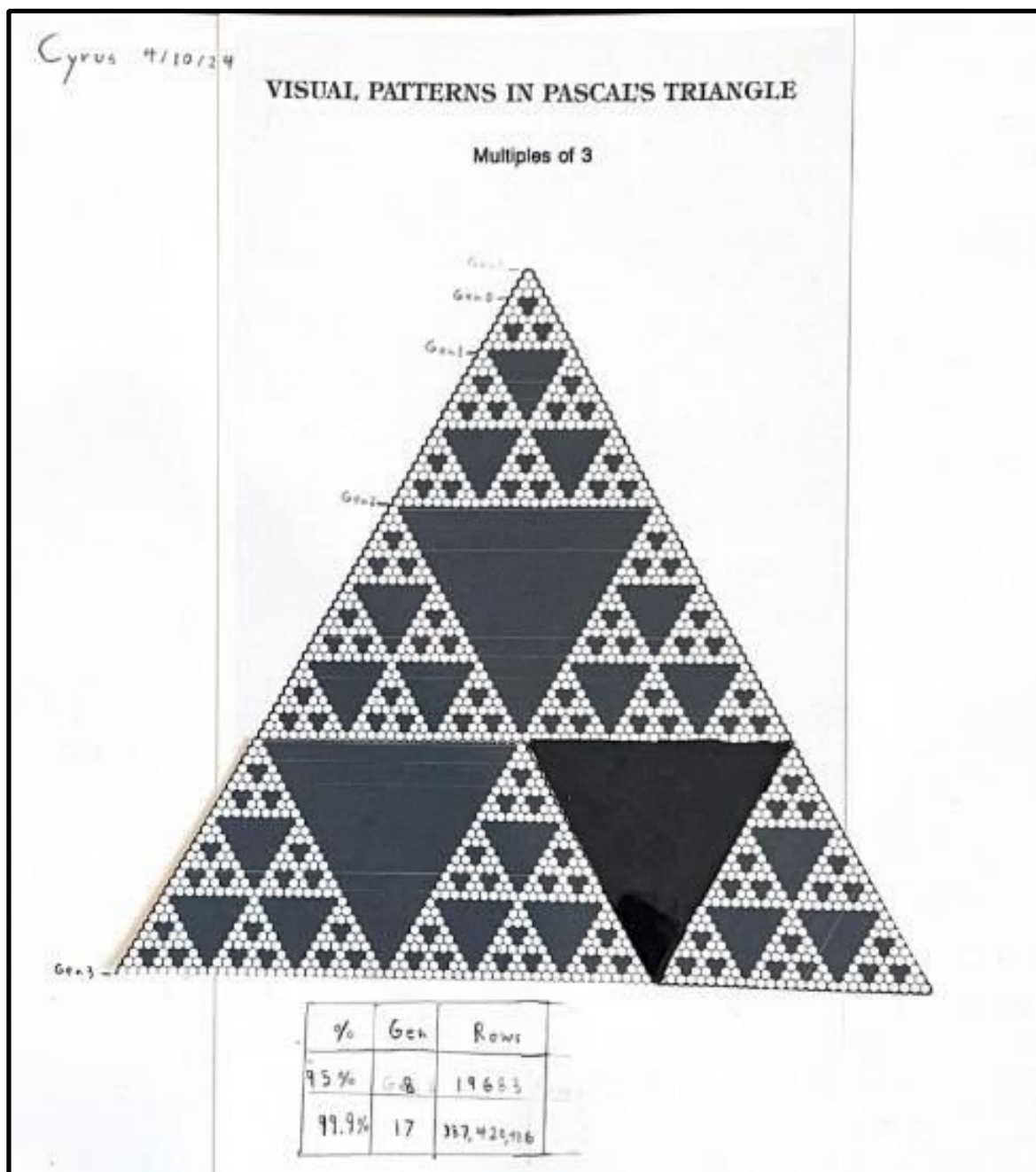
Now I asked them to find the coordinates for B using the Euler's Formula starting from the vector OA. They both recognized that OB is OA rotated by 90 degrees ($\pi/2$ radians) and they also remembered that that would mean multiplying by OA by i . I had to help them with the fact that $i = e^{i\pi/2}$. They needed a little hand holding but Marc got the answer first and corrected the point that I had marked as $2-2i$! (Which I of course said that I had done it wrong to see if they could figure it out! We always get an angry laugh out of this comment). Look at Marc's writing, a very incredible improvement. He is really trying. Cyrus also did it in his methodical way and said that he does not get it at least twice! But they both did the work and had an OK understanding of what they were doing I believe. I will have another example in the next session so the equivalence of the coordinates in the complex plane and their derivation by the Euler's Formula stays with them.



My notes for guiding them.

$$\begin{aligned} e^{i\frac{\pi}{2}} \cdot \vec{OA} &= 2\sqrt{2} e^{i\frac{\pi}{4}} \cdot e^{i\frac{\pi}{2}} \\ &= 2\sqrt{2} e^{i(\frac{\pi}{2} + \frac{\pi}{4})} \\ &= 2\sqrt{2} \left[\cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right) \right] \\ &= 2\sqrt{2} \left[-\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2} \right] \\ &= -2 + 2i \end{aligned}$$

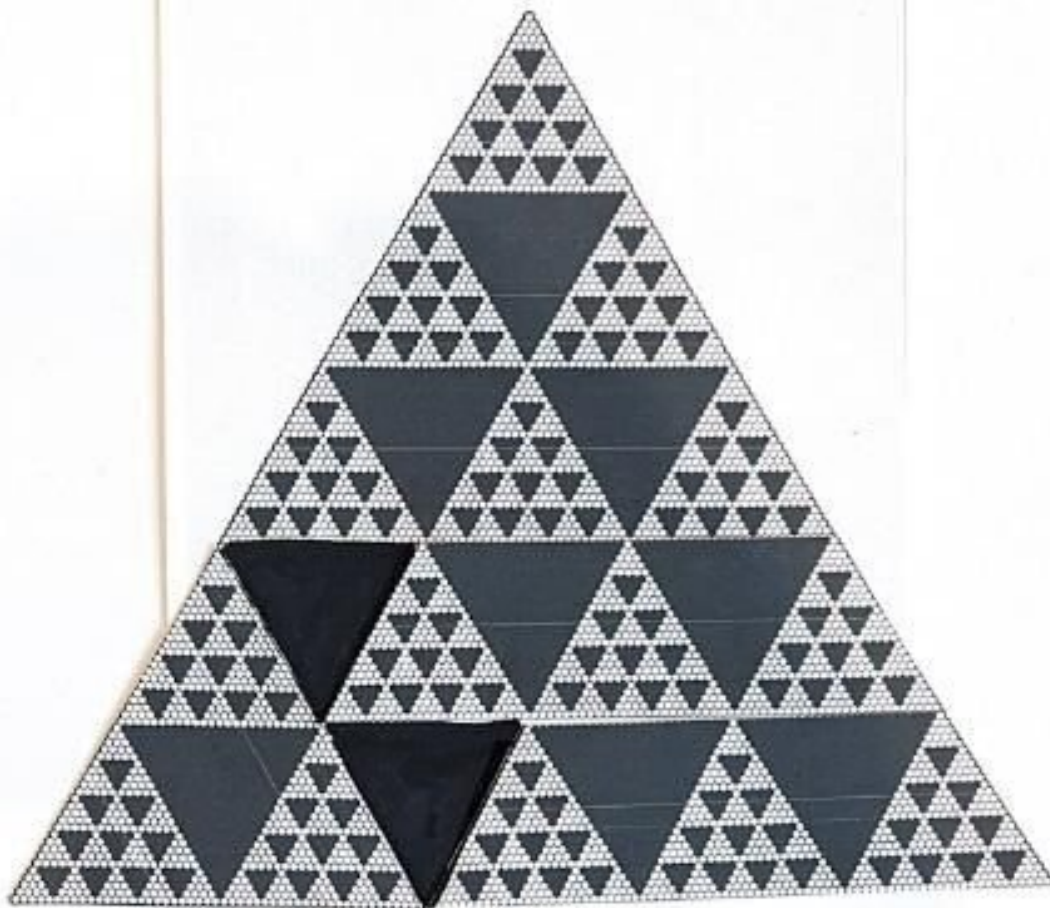
We then went back to the Visual Patterns for Pascal's Triangle for cells that were divisible by 3 and by 5. I had cut out some more of the Triangle so they could at least see 4 generations for 3 that is 3^4 rows and 3 generations for 5 that is 5^3 . They liked this work even though Cyrus complained jokingly if they really had to do it. Marc was on it and completed both quickly. Next I had them examine the tables and charts that would show the number of generation and the number of rows necessary to have 95% and 99.9% divisibility for 2, 3, 5, 7, 11, and 13. We also discuss Fractal's and how these were deterministic Fractals that repeated exactly and how in nature a lot of stuff were Fractals but with a Random component like a cauliflower or a fern and I told them that even the liver and lung tissues were Fractals.



Cyrus 4/10/24

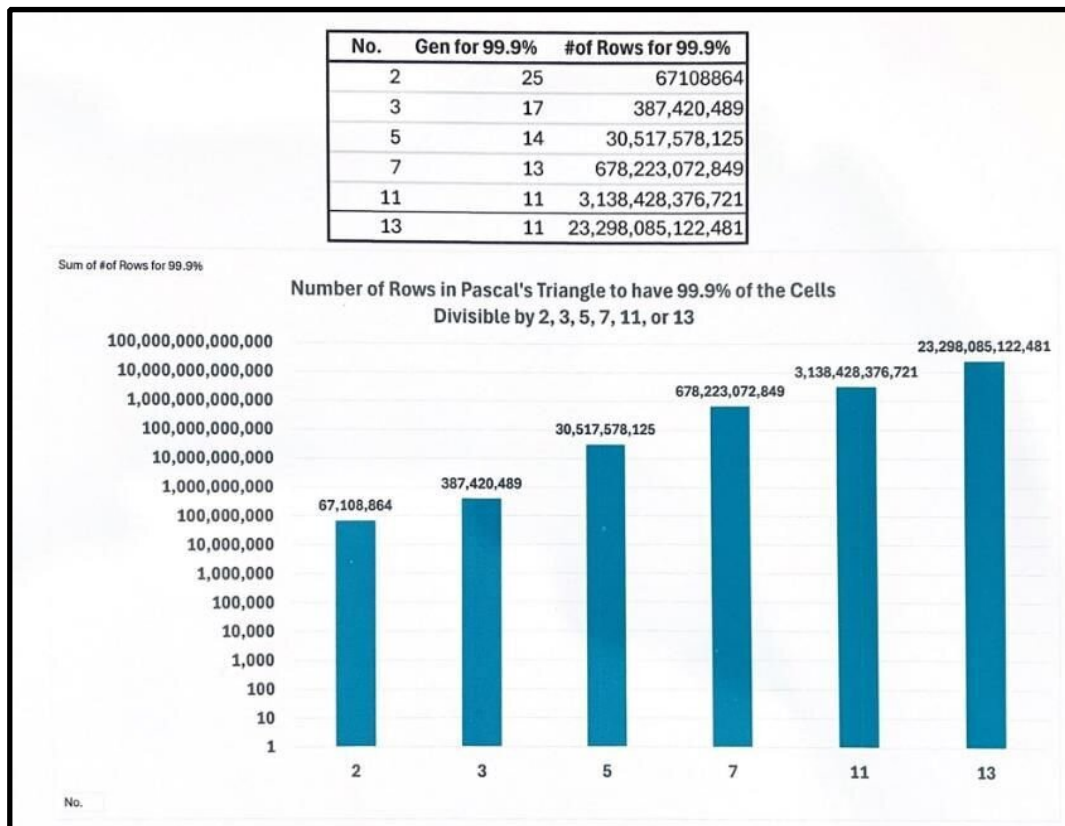
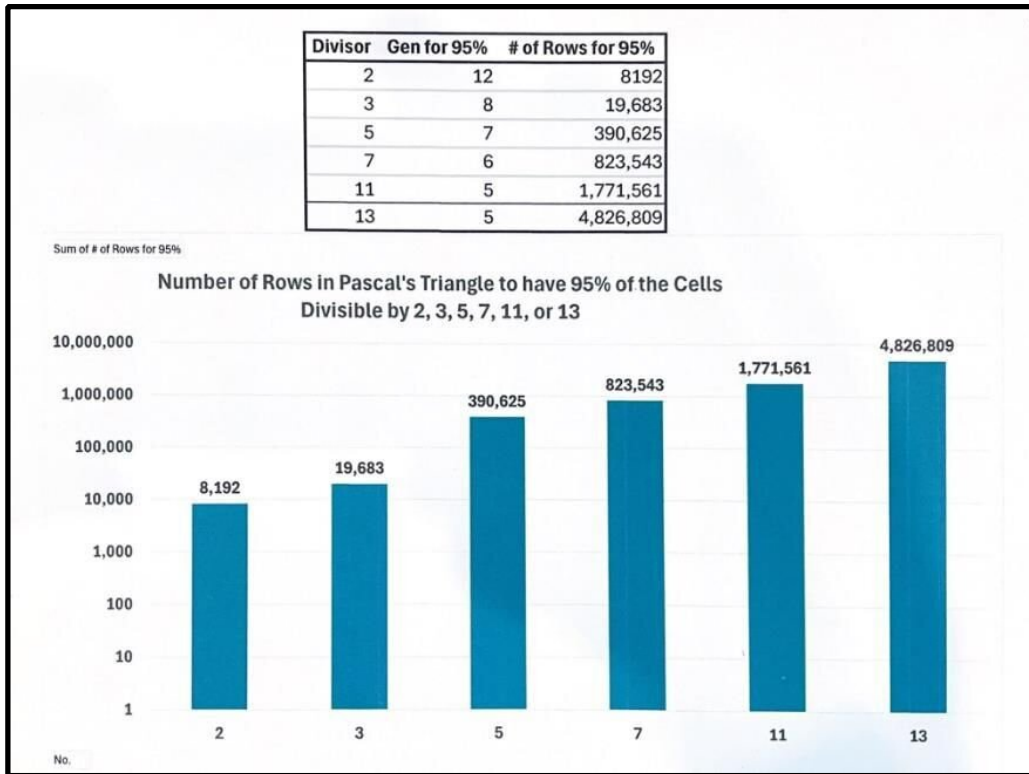
VISUAL PATTERNS IN PASCAL'S TRIANGLE

Multiples of 5



%	Gen	Rows
95%	7	390,625
99.9%	14	30,517,578,125

They used these charts to make tables (the tables were their idea) for 3 and 5 and to see that for 99.9 % we needed an enormous number of rows. Mark was going fast again and got a little sloppy;-<



They also examined the divisibility by 7 from the book Visual Patterns in Pascal's Triangle and I showed them the error that was made for 99.9% divisibility.

CHART 12

MULTIPLES OF SEVEN IN PASCAL'S TRIANGLE

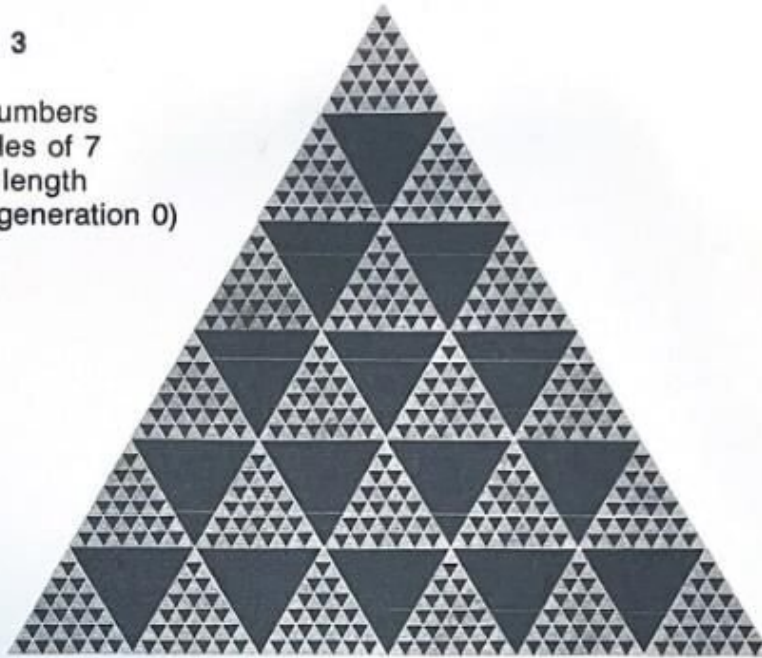
Generation 3

2401 rows*

2,883,601 numbers

79% multiples of 7

(36 ft edge length
at scale of generation 0)



Generation 4 would have 16,807 rows*; 141,246,028 numbers; 88% multiples of 7; (252 ft edge length at scale of generation 0).

Generation 5 would have 117,649 rows*; 6,920,702,425 numbers; 93% multiples of 7; (1765 ft edge length at scale of generation 0).

Generation 6 would have 823,543 rows*; 1.1980×10^{12} numbers; 99.9% multiples of 7; (2.3 mi edge length at scale of generation 0).

96%

*Counted here as actual rows as opposed to numbering system in Pascal's Triangle, which counts the first row as row 0.

Next, they were getting tired but I did show them in Excel how easy it was to actually for the percent divisibility for any prime number. Finally, Cyrus choose 101 and I showed them with a couple of replace commands how they had the results and how large the number of cells was for 99.9% divisibility.

Total Cells	Total Div 101	Gen	PCT Div101
5,151	0	0	0.0%
52,035,301	25,502,500	1	49.0%
530,760,590,451	394,090,132,500	2	74.3%
5,414,283,580,170,600	4,710,294,051,265,000	3	87.0%
55,231,106,275,815,300,000	51,604,856,212,422,500,000	4	93.4%
563,412,515,066,516,000,000,000	544,733,700,989,980,000,000,000	5	96.7%
5,747,371,066,188,170,000,000,000,000	5,651,156,494,879,930,000,000,000,000	6	98.3%
58,628,932,246,184,900,000,000,000,000,000	58,133,330,989,376,200,000,000,000,000,000	7	99.2%
598,073,737,843,332,000,000,000,000,000,000,000	595,520,895,769,511,000,000,000,000,000,000,000	8	99.6%
6,100,950,199,739,830,000,000,000,000,000,000,000,000	6,087,800,510,217,580,000,000,000,000,000,000,000,000	9	99.8%
62,235,792,987,546,000,000,000,000,000,000,000,000,000,000	62,168,058,936,816,900,000,000,000,000,000,000,000,000,000	10	99.9%

They got a good sense that even though almost all the cells in the Pascal's Triangle will be divisible by Any Prime Number, you may have to go to a very large number of rows in the Pascal's Triangle to get there.

I had prepared a preview page for derivatives and wanted them to watch the third video in the Calculus series from 3 Blue 1 Brown, but they were done and wanted to play Foosball, so we went and played and Hugo and Laura had arrived also. They also told me that they wanted to program the next session. So, I have my work cut out for me for the next session.

This was also a great session and the feeling that they are discovering a new concept that nobody else knows about is as exciting to them as it is to me!

Find the letters and mark both representations of these vectors in the complex plane below.

Complex plane diagram showing the Real Axis and imaginary Axis. A unit circle is centered at the origin. Vectors are drawn from the origin to various points labeled with letters. The points are: A at (1, 0), B at (1/2, $\sqrt{3}/2$), C at (2, 2i), D at ($-\frac{\sqrt{2}}{2}$, $-i(\frac{\sqrt{2}}{2})$), E at (-1, 0), F at (-2, -2i), G at ($-\frac{1}{2}$, $-i(\frac{\sqrt{3}}{2})$), H at ($\frac{\sqrt{2}}{2}$, $i(\frac{\sqrt{2}}{2})$), I at (-2, 2i), J at ($\frac{\sqrt{2}}{2}$, $i(\frac{\sqrt{2}}{2})$), K at $2\sqrt{2} \exp(i\pi/4)$, L at $\exp(i\pi/8)$, M at $\exp(i\pi/4)$, N at $2\sqrt{2} \exp(i(\pi/4 + \pi/4))$, O at $\exp(i(3\pi/4))$, P at $2\sqrt{2} \exp(i(\pi/2 + \pi/4))$, Q at $\exp(i(\pi + \pi/4))$, R at $\exp(i\pi/4)$, S at $\exp(-i\pi/4)$, and T at $\exp(0)$. The angle θ is marked between the positive real axis and the vector to point H. The formula $e^{i\theta} = \cos\theta + i \sin\theta$ is written near the unit circle.

Real Axis

imaginary Axis

$e^{i\theta} = \cos\theta + i \sin\theta$

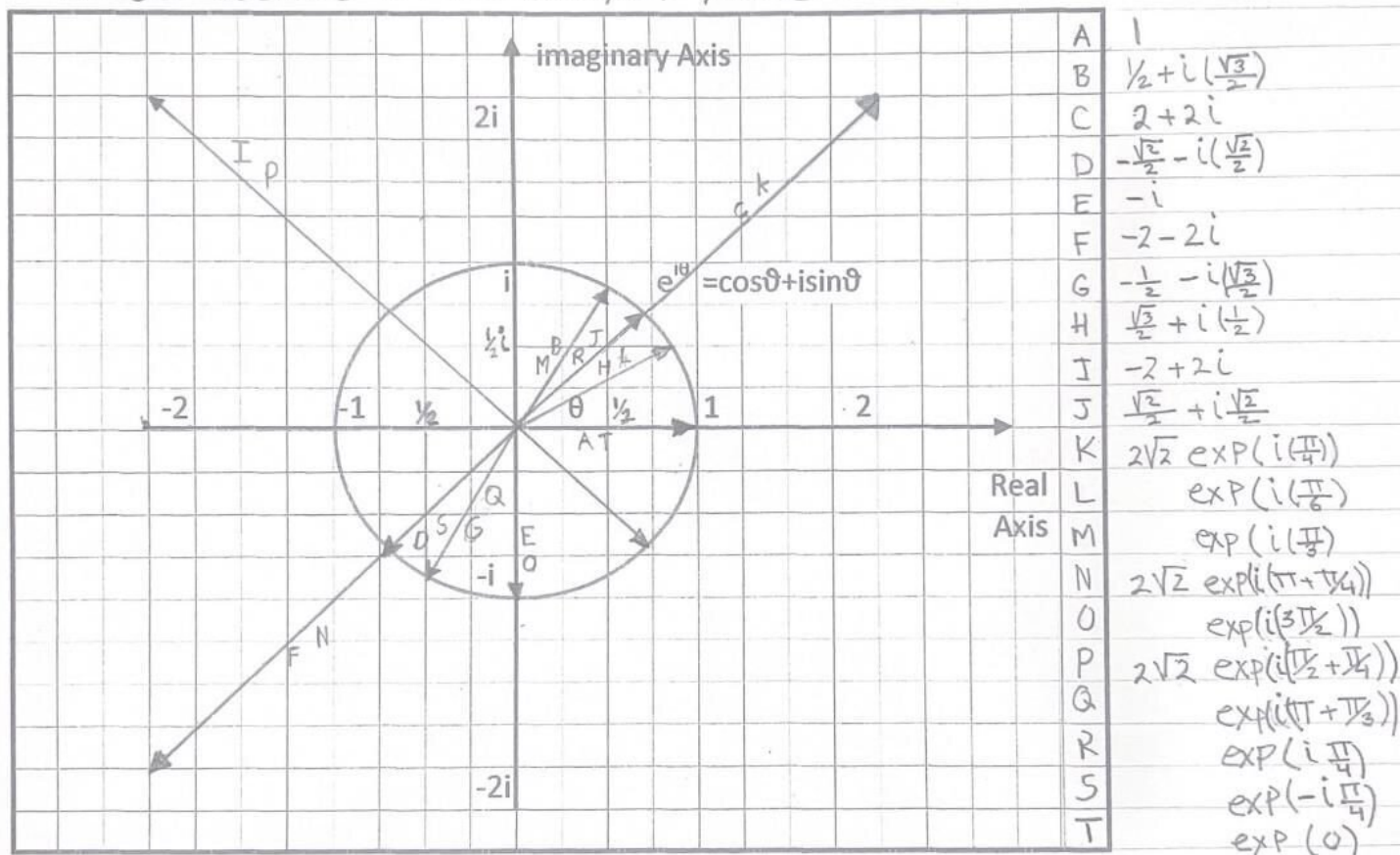
Points labeled: A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T.

Formulas for points:

- A: 1
- B: $\frac{1}{2} + i(\frac{\sqrt{3}}{2})$
- C: $2 + 2i$
- D: $-\frac{\sqrt{2}}{2} - i(\frac{\sqrt{2}}{2})$
- E: -1
- F: $-2 - 2i$
- G: $-\frac{1}{2} - i(\frac{\sqrt{3}}{2})$
- H: $\frac{\sqrt{2}}{2} + i(\frac{\sqrt{2}}{2})$
- I: $-2 + 2i$
- J: $\frac{\sqrt{2}}{2} + i(\frac{\sqrt{2}}{2})$
- K: $2\sqrt{2} \exp(i\pi/4)$
- L: $\exp(i\pi/8)$
- M: $\exp(i\pi/4)$
- N: $2\sqrt{2} \exp(i(\pi/4 + \pi/4))$
- O: $\exp(i(3\pi/4))$
- P: $2\sqrt{2} \exp(i(\pi/2 + \pi/4))$
- Q: $\exp(i(\pi + \pi/4))$
- R: $\exp(i\pi/4)$
- S: $\exp(-i\pi/4)$
- T: $\exp(0)$

Equation: $e^{ix} = \exp(ix)$

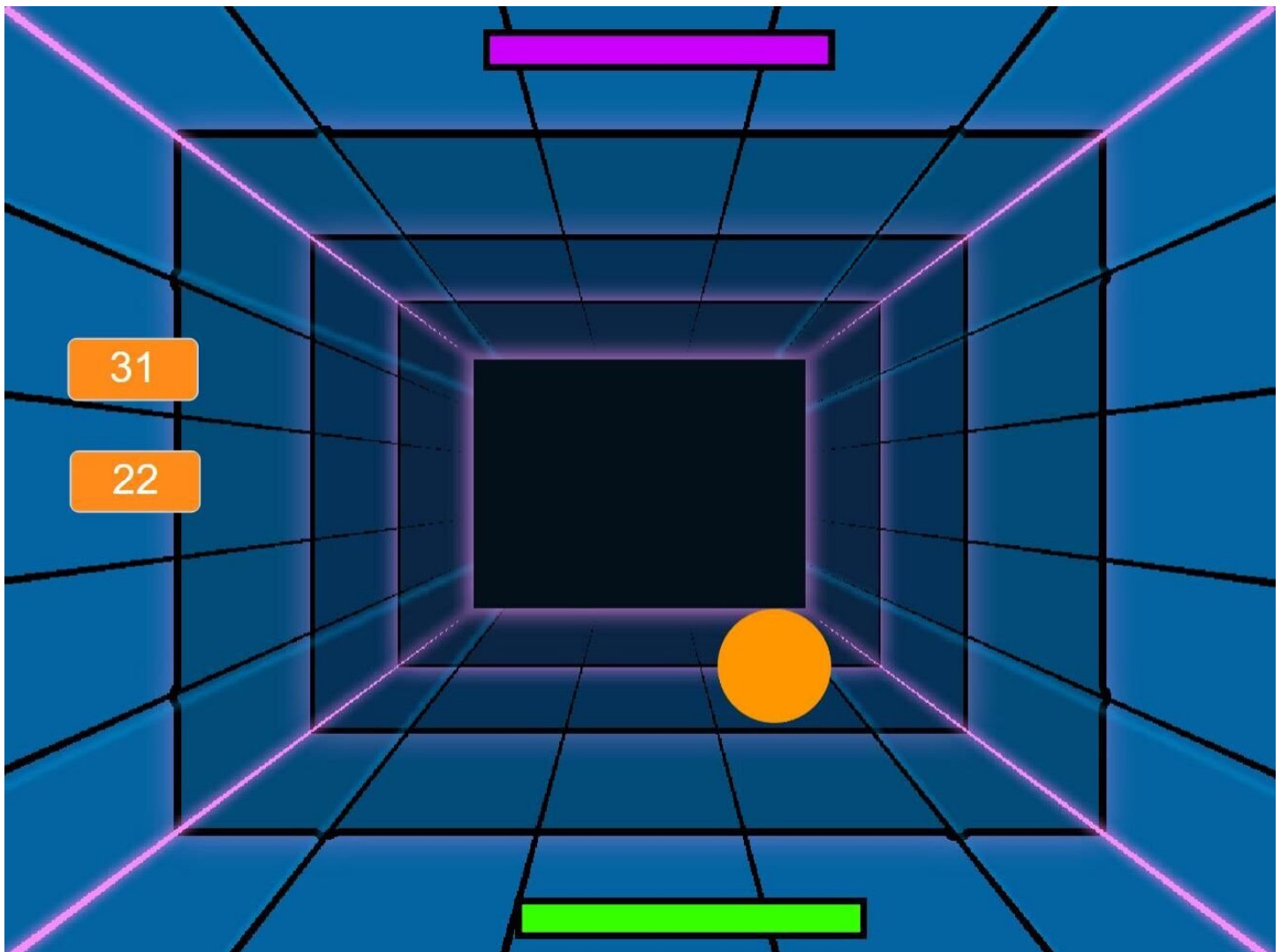
Find the letters and mark both representations of these vectors in the complex plane below.

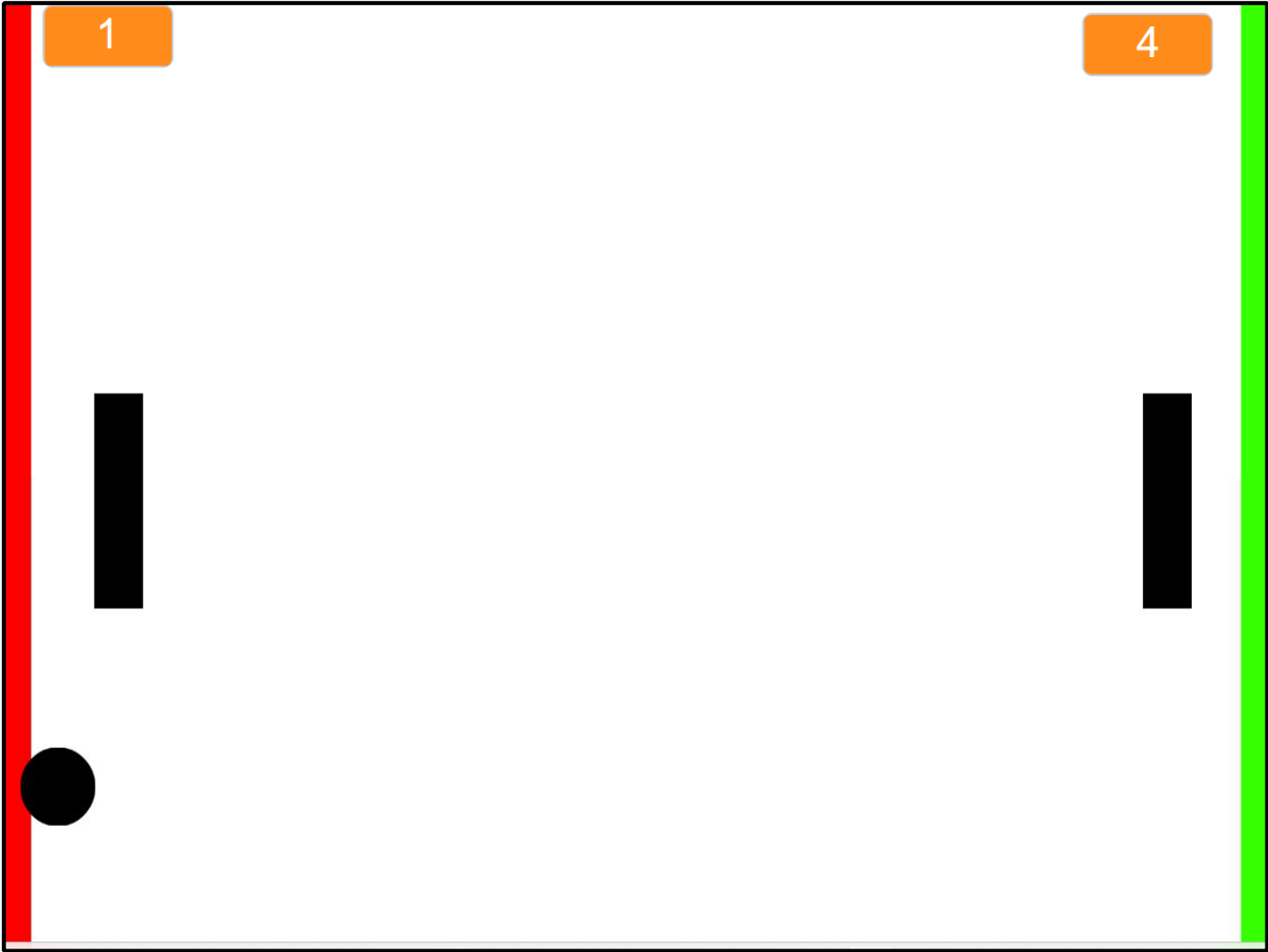


Next, we got to program in Scratch that they have asked for some time now. I had to go and work on my Scratch to be able to guide them. I picked the game of Pong that is one of the simplest to program. They both did well. Marc had already done a little programming in Scratch and Cyrus had played with Scratch without doing any real programming. They are both comfortable with the ideas of variable and reason well.

Marc was up and running and I helped Cyrus more, but they put in a good hour and did not even ask to stop and go play Foosball or anything. I forwarded Marc's project to you already, and Cyrus has worked on his project a couple more times. I was very impressed by how easily they understood the concepts of events and controls and forever loops, bouncing off the walls and paddles and how to set up points for winning or losing in addition to initializing and using variables in the program for the position of the ball and paddles etc. They are also very familiar with the x-y plane so could easily reason about how to use these coordinates for the paddle and the ball and for scoring. The initial game for both had only one paddle but I worked with Cyrus to add another paddle, and scoring with some help from me. Marc had done this himself.

This was also a very good session and we should encourage them to do more programming in the summer.





Math Circle 20 Logarithms 4-24-2024 1:50-2:50

Cyrus, Marc and Dar

We worked on Logarithms. I decided to give them the exercises from the [Logarithm Fundamentals Video from 3 Blue one Brown Live](#). As the live goes too slow for them I decided to do the explanations myself and made cards for them with the questions.

I basically wanted to do what Grant did in the video and gave the same questions to them I also added some myself. Starting with the basic fact that in base 10 Logarithms count zeros when you evaluate the powers of 10. I had already mentioned that I called Logarithms the “Power Spitter Functions” or simply “Spitters” and they liked my name better. Also spitting can be done for all bases for the powers of the base so it is more general. We discussed this for a little.

I had made all the questions from the live into cards. First, they answered that they “have learned about them but they were confused about them”. Next, we went over the rules and they used that card over and over to answer the other questions. The last rule was the hardest for them to digest.

Card 1: What best describes your relationship with logarithms?

- ☐ I have never learned about them.
- ☒ I've learned about them, but sometimes get confused by all the properties.
- ☐ I understand them, but wouldn't know how to teach them.
- ☐ I understand them well and could comfortably teach someone else to understand them well too.

Card 2: Which log rule is the worst?

- ☐ $\log(ab) = \log(a) + \log(b)$
- ☐ $\log(a^n) = n \log(a)$
- ☐ $\log_a(a) = \frac{1}{\log_a(b)}$
- ☒ $\log_a(a) = \frac{\log(b)}{\log(a)}$

Card 3: Which of the following is true?

- ☐ $\log(1,000 \cdot x) = 3 + \log(x)$
- ☐ $\log(1,000 \cdot x) = 3 \cdot \log(x)$
- ☐ $\log(1,000 \cdot x) = 3 + \log(x)$
- ☐ None of the above.

Card 4: Which of the following is true?

- ☐ $\log_a(b) = -\log_a(a)$
- ☒ $\log_a(b) = 1 / \log_a(a)$
- ☐ $\log_a(b) = 1 - \log_a(a)$
- ☐ $\log_a(a) = \log_a(1/b)$
- ☐ None of the above

Card 5: Which of the following is true?

- ☒ $\log(x^n) = n \log(x)$
- ☒ $\log(x^n) = (\log(x))^n$
- ☒ $\log(x^n) = n + \log(x)$
- ☒ $\log(x^n) = \log(n) \cdot \log(x)$
- ☒ None of the above

Card 6: Which of the following is true?

- ☒ $\log(a + b) = \log(a) + \log(b)$
- ☒ $\log(a \cdot b) = \log(a) \log(b)$
- ☐ $\log(a + b) = \frac{1}{\log(a) + \log(b)}$
- ☐ $\log(a + b) = \frac{1}{\log(a) \log(b)}$
- ☐ None of the above

Card 7: Given the fact that $2^{10} \approx 1,024$, which is approximately 1,000, which of the following is closest to being true?

- ☒ $\log_2(10) \approx 0.3$
- ☒ $\log_{10}(2) \approx 0.3$
- ☐ $\log_2(10) \approx 1/3$
- ☐ $\log_{10}(2) \approx 1/3$

Handwritten Notes:

- Logarithm values:
 - $\log_{10}(1) = 0$
 - $\log_{10}(10) = 1$
 - $\log_{10}(100) = 2$
 - $\log_{10}(1000) = 3$
 - $\log_{10}(10,000) = 4$
 - $\log_{10}(100,000) = 5$
 - $\log_{10}(1,000,000) = 6$
 - $\log_{10}(10^9) = 9$
 - $\log_{10}(10^{12}) = 12$
 - $\log_{10}(10^n) = n$
 - $\log_2(2^n) = n$
 - $\log_a(a^n) = n$
- Triangle diagram with vertices 10, 1,000, and 3.
- Calculations:
 - $10^3 = 1,000$
 - $\sqrt[3]{1,000} = 10$
 - $\log_{10}(1,000) = 3$

Which log rule is the worst?

☐ $\log(ab) = \log(a) + \log(b)$

☐ $\log(a^n) = n \log(a)$

☐ $\log_b(a) = \frac{1}{\log_a(b)}$

☐ $\log_b(a) = \frac{\log_c(b)}{\log_c(a)}$

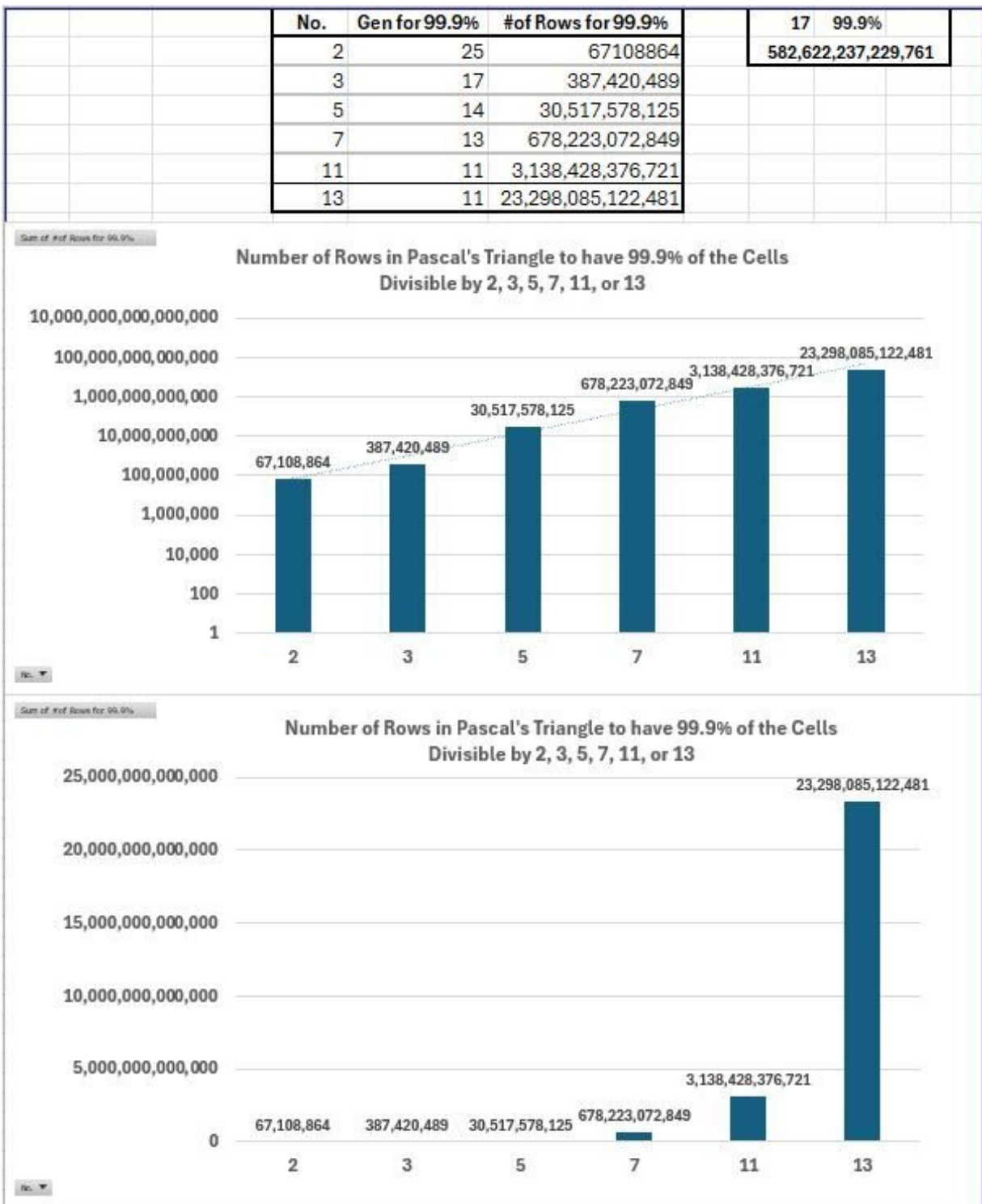
$$\log(ab) = \log(a) + \log(b)$$

$$\log(a^n) = n \log(a)$$

$$\log_b(a) = 1/\log_a(b)$$

$$\log_b(a) = \log_c(a)/\log_c(b)$$

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Magnitude	Description	Typical maximum modified Mercalli intensity ^[11]	Average earthquake effects	Average frequency of occurrence globally (estimated)
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It was a good session and Cyrus wanted to know when we were going to do natural logs (in base e) that I had mentioned to him before so I said we will look at that next time.

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What is the following?

$$\frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \dots + \frac{1}{\log_{100}(100!)}?$$

Enter your answer

Cyrus, Marc and Dar

We continued the work on Logarithms. I also continued to give them the exercises from the [Logarithm Fundamentals Video from 3 Blue one Brown Live](#). As the live goes too slow for them I decided to do the explanations myself and made cards for them with the questions. We also watched part of the Math when used Desmos graphic calculator.

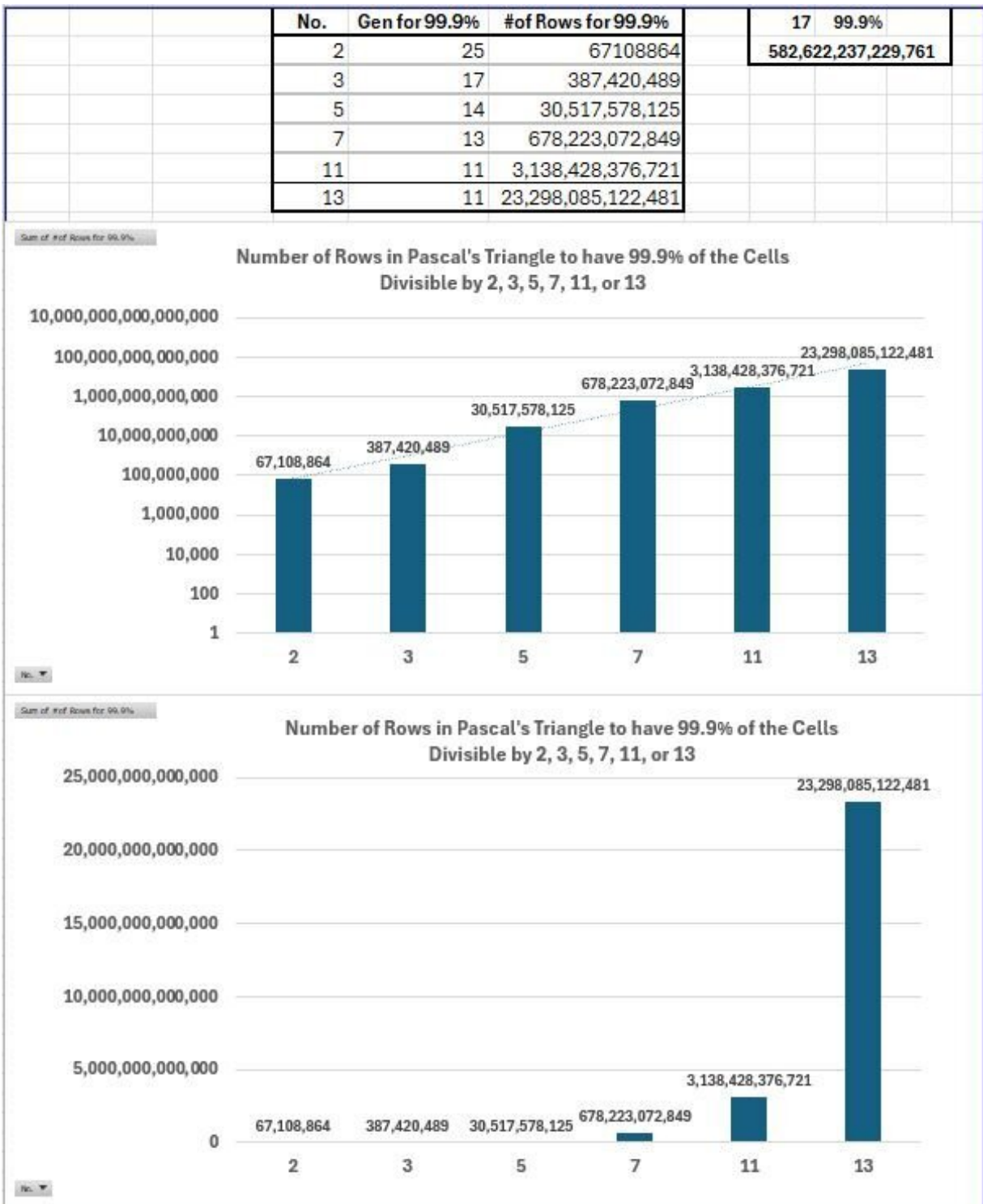
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Enter your answer



$10^3 = 1,000$
 $\quad$
 $3\sqrt[3]{1,000} = 10$
 $\quad$
 $\log_{10}(1,000) = 3$

Math Circle 22 Logarithms and Exponentials Final 5-08-2024 1:55-3:00

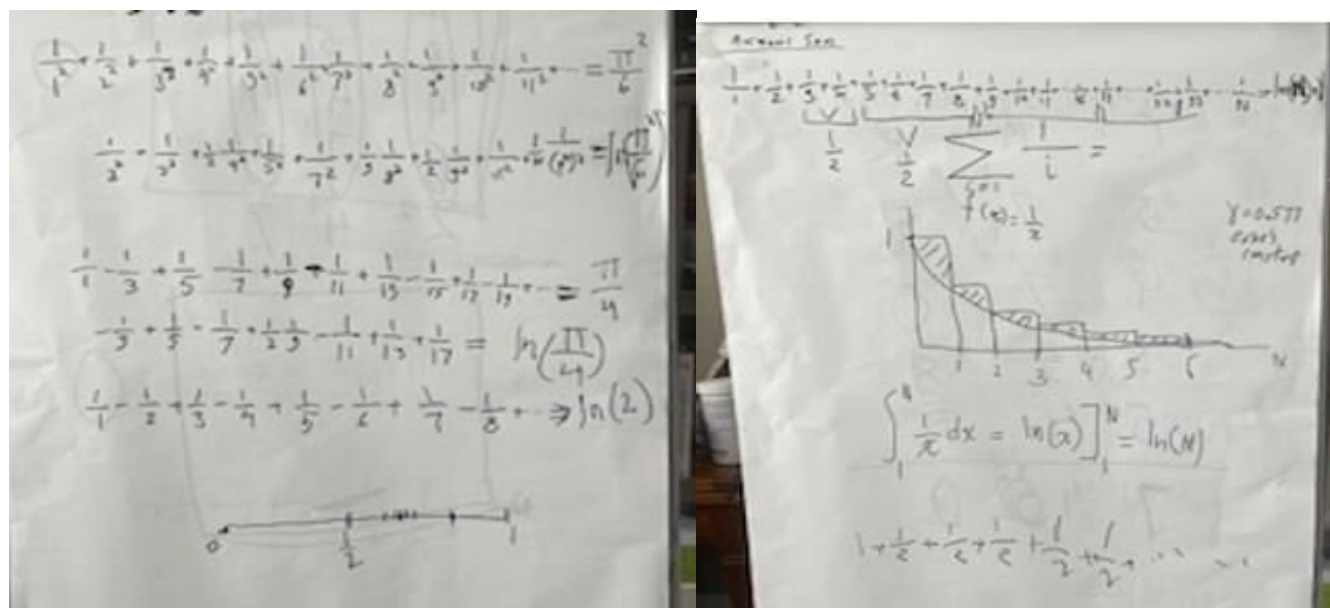
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We continued the work on Logarithms. I also continued to give them the exercises from the [Logarithm Fundamentals Video from 3 Blue one Brown Live](#). As before I did the explanations myself and made cards for them with the questions.

Below is what I have put on the Board to help them get through the questions. First I discussed again how strange it is that $\pi^2/6$ and $\pi/4$ appear as the limit of these sequences and what is even more remarkable is that when we manipulate these series we get $\ln(\pi^2/6)$ and $\ln(\pi/4)$!

They were not as fascinated by this as I was! I do not blame them;=> Next we looked at the last series and how it will converge to $\ln(2)$ almost equal to 0.69 and how $\ln()$ keeps popping up in these strange place. Cyrus kept asking why it is called the natural log and I told them because it shows up in these strange places!

Next, I gave them the question about the proportion of primes between 1,000,000,000,000 and 1,000,000,001,000 they both guessed 1/250 and I showed them that this is really more like 1/25 using the $\ln(1,000,000,001,000)$. This is another reason why $\ln()$ is called the natural log.



Next I gave them the sheet to compare the Harmonic series with the area under the curve of $1/x$. I did go through a lot of explaining that the integral is the area under the curve and the integral of $1/x$ is $\ln(x)$. I told them they just had to accept this for now. I helped them draw a histogram for the Harmonic series and also put the function $1/x$ on top of it and see that these two areas differ by a constant area at the limit which is called the Euler's Constant γ (gamma). I had to help them a lot, but they got through it. I had prepared a sheet for them and had done some of the work there so they could complete it as this is very full of complicated concepts for them.

Consider the numbers between 1,000,000,000,000 and 1,000,000,001,000. That is, a range of one thousand integers starting at one trillion.

Which of the following would you guess is closest to the proportion of these numbers that are prime?

- ☒ 1/10
- ☒ 1/25
- ☒ 1/100
- ☒ 1/250
- ☒ 1/1,000

$\ln(N)$
Density of primes near N

13

Let $S = \sum_{n=1}^N \frac{1}{n}$ and $I = \int_1^N \frac{dx}{x}$. Which of the following is true?

- ☐ $S < I$
- ☐ $S = I$
- ☒ $S > I$

derivative of $\ln(x) = 1/x$
 $\int \boxed{1/x} dx = \ln(x)$

18

Which of the following is closest to the smallest value of N for which the sum $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N}$ is larger than 1,000,000?

- ☐ 10^6
- ☐ 10^{144}
- ☐ $10^{6,685}$
- ☒ $10^{434,294}$
- ☐ $10^{1,000,000}$

$$e^x = 10^x \quad e = 10^{\frac{1}{x}}$$

$$x \ln e = \ln 10$$

$$x = \ln 10 \approx 2.3 \approx 2$$

$$e = 10^{\frac{1}{2}}$$

14

Prove that the derivative of e^x is equal to e^x

Use the Taylor series expansion for

$$e^x = 1 + x^2/2! + x^3/3! + x^4/4! + \dots + x^n/n! + \dots$$

Use the Power Rule for derivatives:

Derivative of $x^n = nx^{(n-1)}$ e.g. derivative of $x^4 = 4x^3$

$$\frac{d(e^x)}{dx} = 0 + 1 + \frac{x^2}{2} + \frac{x^3}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

16

They also found that 10^{434294} was the answer to problem 14. Finally, I helped them see that derivative of e^x is equal to e^x . I had told them about the derivative of the powers of x but I reviewed it and they did figure it out and were mildly impressed by it. Marc saw it first and in typical fashion Cyrus thought about it and agreed.

This was a difficult session, but they did well, and I am sure we will come back to these topics next year. Marc was tired but he held on until the end of the session.

$$log_2(x) = 1,000,000 \quad x = 2^{1,000,000}$$

$$n(H) = 1,000,000$$

$$N = 1,000,000 \rightarrow 10^6$$

$$e^2 = 10 \quad e = 10^{\frac{1}{2}}$$

$$2^{1/5} = 1.1 \quad x = 1.1$$

$$e \in \{2, 3\}$$

$$\frac{1}{\log(100!)} = \frac{\log 2}{\log 100} + \frac{\log 3}{\log 100} + \frac{\log 4}{\log 100} + \dots$$

1. $\frac{1}{2} \times 100 = 50$

$$\log 2$$

$$\log_{10} 2 \times 3 \times 4 \times 5 \times 6 = \log_{10} 1000 = 3$$

1
258

$$\ln(10^{12}) = 27.6$$

1
1 2
1 3
1 4
1 5
1 2 3 4

17
1278
139

$$\begin{array}{r} 1000 \\ 9 \quad 8 \\ 10 \quad 9 \\ \hline 9 \quad 8 \quad 5 \quad 4 \end{array}$$

$$\frac{1}{\log_2(1000)} = \frac{\log_2 2}{\log_2(1000)} + \frac{\log_2 2^2}{\log_2(1000)} + \frac{\log_2 2^4}{\log_2(1000)} + \dots$$

[illegible]

Aug 2
(100)

$$\log_{10} 100 = 2, \quad \log_{10} 1000 = 3$$

154

$$\ln(10^{12}) = 2.6$$

$$\begin{array}{l} 1 \quad 2 \\ 1 \quad 3 \\ 1 \quad 2 \quad 4 \\ 1 \quad 5 \\ 1 \quad 2 \quad 3 \quad 6 \end{array}$$

17
1248
152

$$\begin{array}{r} 1000 \\ 9 \quad 8 \\ 10 \quad 7 \\ \hline 17 \end{array}$$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

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derivative of $e^x = e^x$

$$\dim_{\mathbb{R}} X^{\otimes n} = n \cdot \dim_{\mathbb{R}} X$$

Harmonic Series

Natural and Logarithms (ln)

$$A_N = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{N} = \ln(N) + \gamma$$

$$B_N = \int_1^N \frac{1}{x} dx = \ln(x) \Big|_1^N = \ln(N)$$

$$f(x) = \frac{1}{x}$$

Graph A_N and B_N

What is true

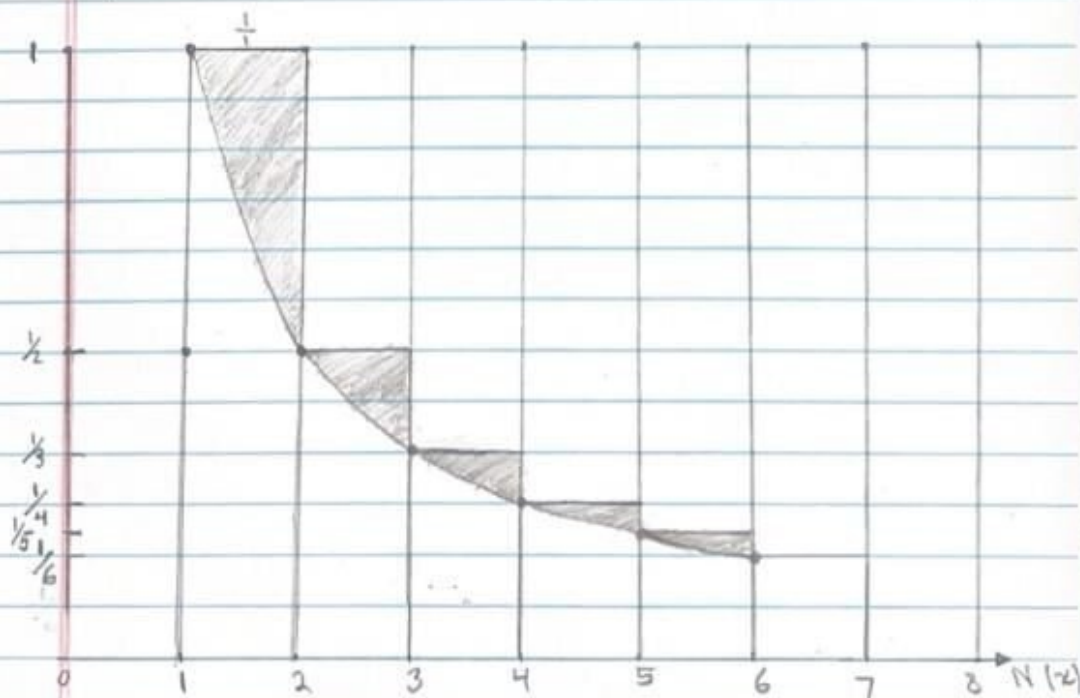
☐ $A_N < B_N$

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x	$\frac{1}{x}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$
5	$\frac{1}{5}$
6	$\frac{1}{6}$

γ = Euler's Constant



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16

Derivatives & Integrals

The power rule

the derivative of 1 is 0 the integral

$$\frac{d(1)}{dx} = 0$$

the derivative of x is 1 the integral of 1 is x

$$\frac{d(x)}{dx} = 1$$

$$\int 1 dx = x$$

function	derivative	Integral
1	0	x
x	1	$\frac{x^2}{2}$
x^2	$2x$	$\frac{x^3}{3}$
x^3	$3x^2$	$\frac{x^4}{4}$
x^n	nx^{n-1}	$\frac{x^{n+1}}{n+1}$

derivatives are slopes (rates of change)

Integrals are area under a curve

derivative

function	derivative	Integral
$\log x$	$\frac{1}{x}$	$x \log x - x$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\log x$
$\sin x$	$\cos x$	$-\cos x$
$\cos x$	$-\sin x$	$\sin x$

Chain Rule derivative of $f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Responses for the first week of logarithms:

1

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4

Which of the following is true?

- ☐ $\log(1,000 \cdot x) = 2 \log(x)$
- ☐ $\log(1,000 \cdot x) = \log(x)$
- ☒ $\log(1,000 \cdot x) = 3 + \log(x)$
- ☐ $\log(1,000 \cdot x) = 3$
- ☐ None of the above

7

Which of the following is true?

- ☐ $\log_2(8) = -\log_2(x)$
- ☒ $\log_2(8) = 3 / \log_2(x)$
- ☐ $\log_2(8) = 1 - \log_2(x)$
- ☐ $\log_2(x) = \log_2(1/8)$
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- ☐ $\log(a^n) = n \log(a)$
- ☐ $\log_a(x) = \frac{1}{\log_b(x)}$
- ☒ $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

5

Which of the following is true?

- ☒ $\log(x^n) = n \log(x)$
- ☐ $\log(x^n) = \log(x)^n$
- ☐ $\log(x^n) = n + \log(x)$
- ☐ $\log(x^n) = \log(x) \cdot \log(x)$
- ☐ None of the above

6

Handwritten notes and diagrams:

$10^3 = 1,000$
 $\downarrow 3$

$\sqrt[3]{1,000} = 10$
 $\downarrow 3$

$\log_{10}(1,000) = 3$

Given the fact that $2^{10} \approx 1,024$, which is approximately 1,000, which of the following is closest to being true?

- ☒ $\log_2(10) \approx 0.3$
- ☒ $\log_{10}(2) \approx 0.3$
- ☐ $\log_2(10) \approx 1/3$
- ☐ $\log_{10}(2) \approx 1/3$

Handwritten calculations:

$$\log_c(b) \cdot \log_b(a) = \log_c(a)$$

$$\log_2(10) \cdot \log_{10}(2) = \log_2(2)$$

$$\log_2(10) \cdot \log_{10}(2) = 1$$

$$\log_2(10) = \frac{1}{\log_{10}(2)}$$

$$\log_2(10) = \frac{1}{\frac{1}{3}} = 3$$

$$\log_2(1000) = \frac{10}{3} \log_{10}(1000)$$

$$10 = \frac{10}{3} \cdot 3 = 10 \checkmark$$

Responses for the Second week of Logarithms:

10

Which of the following is true?

☒ $\log_2(8) + \log_2(x) = \log_2(x)$

☒ $\log_2(8) + \log_2(x) = \log_2(x)$

☒ $\log_2(8) + \log_2(x) = \log_2(x)$

☒ $\log_2(x) + \log_2(8) = \log_2(x)$

☒ None of the above

11

Using the approximation $\log_2(10) \approx \frac{10}{3}$, which of the following is (approximately) true?

☒ $\log_2(x) = \frac{10}{3} + \log_{10}(x)$

☐ $\log_2(x) = \log_{10}\left(\frac{10}{3} \cdot x\right)$

☒ $\log_2(x) = (\log_{10}(x))^{\frac{10}{3}}$

☒ $\log_2(x) = \frac{10}{3} \cdot \log_{10}(x)$

☐ None of the above

12

What is the following?

$$\frac{1}{\log_2(100)} + \frac{1}{\log_4(100)} + \dots + \frac{1}{\log_{100}(100)}?$$

Enter your answer:

$\frac{1}{\log_2(100)} = \log_{100}(2)$

$\log_{100}(2) + \log_{100}(3) + \log_{100}(4) + \dots + \log_{100}(100)$

$= \log_{100}(2 \times 3 \times 4 \times \dots \times 100)$

$= \log_{100}(100!) = 1$

15

Consider the numbers between 1,000,000,000,000 and 1,000,000,000,000,000. That is, a range of one thousand integers starting at one trillion.

Which of the following would you guess is closest to the proportion of these numbers that are prime?

☒ 1/10

☒ 1/25

☒ 1/100

☒ 1/250

☒ 1/1,000

$\ln(N)$
Density of primes near N

Find the sum of all the factors of the numbers 1 to 9.

$1+2+3+4+5+6+7+8+9 = 45$

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$x = 1000$

$\log_2(1000) = 10$

$\log_{10}(1000) = 3$

$\frac{10}{3} \times 3 = \frac{30}{3} = 10$

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All the Questions Covered in the last two sessions:

What best describes your relationship with logarithms?

- ☐ I have never learned about them.
- ☐ I've learned about them, but sometimes get confused by all the properties.
- ☐ I understand them, but wouldn't know how to teach them.
- ☐ I understand them well and could comfortably teach someone else to understand them well too.

Which of the following is true?

- ☐ $\log(1,000 \cdot x) = 3 \log(x)$
- ☐ $\log(1,000 \cdot x) = \log(x)^3$
- ☐ $\log(1,000 \cdot x) = 3 + \log(x)$
- ☐ $\log(1,000 \cdot x) = 3^{\log(x)}$
- ☐ None of the above

Which of the following is true?

- ☐ $\log_a(b) = -\log_b(a)$
- ☐ $\log_a(b) = 1/\log_b(a)$
- ☐ $\log_a(b) = 1 - \log_b(a)$
- ☐ $\log_b(a) = \log_a(1/b)$
- ☐ None of the above

Which log rule is the worst?

- ☐ $\log(ab) = \log(a) + \log(b)$
- ☐ $\log(a^n) = n \log(a)$
- ☐ $\log_b(a) = \frac{1}{\log_a(b)}$
- ☐ $\log_b(a) = \frac{\log_c(b)}{\log_c(a)}$

Which of the following is true?

- ☐ $\log(x^n) = n \log(x)$
- ☐ $\log(x^n) = (\log(x))^n$
- ☐ $\log(x^n) = n + \log(x)$
- ☐ $\log(x^n) = \log(n) \cdot \log(x)$
- ☐ None of the above

Which of the following is true?

- ☐ $\log(a + b) = \log(a) + \log(b)$
- ☐ $\log(a + b) = \log(a) \log(b)$
- ☐ $\log(a + b) = \frac{1}{\log(a) + \log(b)}$
- ☐ $\log(a + b) = \frac{1}{\log(a) \log(b)}$
- ☐ None of the above

Given the fact that $2^{10} = 1,024$, which is approximately 1,000, which of the following is closest to being true?

- ☐ $\log_2(10) \approx 0.3$
- ☐ $\log_{10}(2) \approx 0.3$
- ☐ $\log_2(10) \approx 1/3$
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Using the approximation $\log_2(10) \approx \frac{10}{3}$, which of the following is (approximately) true?

- ☐ $\log_2(x) = \frac{10}{3} + \log_{10}(x)$
- ☐ $\log_2(x) = \log_{10}\left(\frac{10}{3} \cdot x\right)$
- ☐ $\log_2(x) = (\log_{10}(x))^{10/3}$
- ☐ $\log_2(x) = \frac{10}{3} \cdot \log_{10}(x)$
- ☐ None of the above

Which of the following is true?

- ☐ $\log_c(b) \cdot \log_c(a) = \log_c(a)$
- ☐ $\log_c(b) \cdot \log_a(a) = \log_c(a)$
- ☐ $\log_a(c) \cdot \log_a(c) = \log_a(a)$
- ☐ $\log_a(c) \cdot \log_a(b) = \log_a(a)$
- ☐ None of the above

What is the following?

$$\frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \dots + \frac{1}{\log_{100}(100!)}?$$

Enter your answer.



$10^3 = 1,000$
 $\quad$
 $\sqrt[3]{1,000} = 10$
 $\quad$
 $\log_{10}(1,000) = 3$

